Complex numbers for numerical analysis

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Not unexpectedly, a number of students forget most of what they learned with respect to complex numbers following a period of time between learning the material and using that material. This is a review of the concepts you should have seen about complex numbers that you will require in this course on numerical analysis.

We define $j = \sqrt{-1}$, so $j^2 = -1$. Additionally, we note that $j^3 = -j$ and $j^4 = 1$, so $j^5 = j^{1001} = j$.

Given a variable x, we can define a linear polynomial $4 + 5x$. Similarly, given $\sqrt{2}$, we have the real number $4 + 5\sqrt{2}$. In the same way, we can define a complex number $4 + 5j$.

If $z = 4 + 5j$, then the "real component" of z is 4, and the "imaginary component" is the real coefficient 5. These are written as $\Re e(z)$ and $\Im m(z)$, respectively. The imaginary component is a real number.

Two complex numbers w and z are equal, written $w = z$, if and only if both the real and imaginary components are equal.

The zero linear polynomial is $0 + 0x$. The value $0 + 0\sqrt{2} = 0$. The "zero" for a complex number is $0 + 0j$.

All complex numbers with a real component greater than zero are said to form the "open right-hand complex plane", while all complex numbers with a real component less than zero are said to form the "open left-hand complex plane." All complex numbers with an imaginary part equal to zero are said to be "real" and all complex numbers with a real part equal to zero are said to be "imaginary."

We will look at complex arithmetic and then the roots of polynomials.

1 Complex arithmetic

We will review complex addition, complex multiplication, division by a real number, the complex conjugate, complex division, and some properties of complex numbers.

1.1 Complex addition

Given a linear polynomial $7 - 4x$, we can add to this $5 + 3x$ to get the polynomial $12 - x$. Similarly, given two real numbers $7 - 4\sqrt{2}$ and $5 + 3\sqrt{2}$, we can add them to get $12 - \sqrt{2}$. Finally, given two complex numbers $7 - 4j$ and $5 + 3j$, adding these two we get $12 - j$.

Thus, $(\alpha + \beta j) + (\gamma + \delta j) = (\alpha + \gamma) + (\beta + \delta)j$. For example, $(3 + 4j) + (-2 + 7j) = 1 + 11j$.

Note that $(\alpha + \beta j) + (0 + 0j) = \alpha + \beta j$, and $w + z = w$ if and only if $z = 0 + 0j$. Also note that $w + z = z + w$, and that if you have the sum of n complex numbers, like the real numbers, it doesn't matter what order you add them in. Thus, complex addition is both "commutative" and "associative".

Observe that given any complex number $z = \alpha + \beta j$, we can define $-z = -\alpha - \beta j$, so that $z + (-z) = 0 + 0j$. This "additive inverse" of z is unique, and the only number that is its own additive inverse is $0 + 0j$, just in the same way that if x is real, $x = -x$ if and only if $x = 0$.

As you may suspect, subtraction is defined similarly: $(\alpha + \beta j) - (\gamma + \delta j) = (\alpha - \gamma) + (\beta - \delta)j$. For example, $(3 + 4j) - (-2 + 7j) = 5 - 3j$. Again, $w - z = 0$ if and only if $w = z$. Note that like the subtraction of real numbers, complex subtraction is generally neither commutative (so in general, $w - z \neq z - w$), nor associative $(w - (z - v) \neq (w - z) - v)$.

1.2 Complex multiplication

Given the linear polynomials $-2 + 5x$ and $4 + 7x$, we can multiply these two using FOIL (first, outside, inside and last) to get the

$$(-2 + 5x)(4 + 7x) = -8 - 14x + 20x + 35x^2 = -8 + 6x + 35x^2.$$

Similarly, you can multiply $-2 + 5\sqrt{2}$ and $4 + 7\sqrt{2}$ to get

$$(-2 + 5\sqrt{2})(4 + 7\sqrt{2}) = -8 - 14\sqrt{2} + 20\sqrt{2} + 35\sqrt{2}^2 = -8 + 6\sqrt{2} + 35 \cdot 2 = 62 + 6\sqrt{2}.$$

Finally, given the two complex numbers $-2 + 5j$ and $4 + 7j$, we can multiply these two to get

$$(-2 + 5j)(4 + 7j) = -8 - 14j + 20j + 35j^2 = -8 + 6j + 35j^2,$$

but $j^2 = -1$, so this equals $-8 + 6j - 35 = -43 + 6j$.

Thus,

$$(\alpha + \beta j)(\gamma + \delta j) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)j.$$

For example, $(3 + 4j)(-2 + 7j) = -6 + 21j - 8j + 28j^2 = -34 + 13j$.

Note that $(\alpha + \beta j)(0 + 0j) = 0 + 0j$ and that $(\alpha + \beta j)(1 + 0j) = \alpha + \beta j$. Also, $wz = 0$ if and only if $w = 0$, $z = 0$ or both are zero. Similarly $wz = w$ if and only if either $w = 0+0j$, $z = 1+0j$, or both. Also note that $wz = zw$, and that if you have the product of n complex numbers, like the real numbers, it doesn't matter what order you multiply them in. Thus, complex multiplication is both "commutative" and "associative".

1.3 Dividing a complex number by a real number

Given a polynomial $-5 - 9x$, we can divide this by a real number by simply dividing both coefficients, so $\frac{-5-9x}{-2.5} = 2 + 3.6x$. Similarly, we can divide the radical $-5 - 9\sqrt{2}$ by that same real number to get $\frac{-5-9\sqrt{2}}{-2.5} = 2 + 3.6\sqrt{2}$. We can do the same for a complex number $-5 - 9j$ and divide it by -2.5 to get $\frac{-5-9j}{-2.5} = 2 + 3.6j$. Clearly, we cannot divide by zero.

Thus, for a real number $\gamma \neq 0$, $\frac{\alpha+\beta j}{\gamma} = \frac{\alpha}{\gamma} + \frac{\beta}{\gamma}j$. For example, $\frac{8-15j}{-6} = -\frac{4}{3} + \frac{5}{2}j$.

1.4 Complex division

Given two polynomials $5 + 7x$ and $-4 + 3x$, we generally cannot simplify the ratio $\frac{5+7x}{-4+3x}$ unless there is a common factor; however, given the radicals $5 + 7\sqrt{2}$ and $-4 + 3\sqrt{2}$, we can simplify this by multiplying by the "radical conjugate" of the denominator over itself:

$$\frac{5 + 7\sqrt{2}}{-4 + 3\sqrt{2}} = \frac{5 + 7\sqrt{2}}{-4 + 3\sqrt{2}} \cdot 1 = \frac{5 + 7\sqrt{2}}{-4 + 3\sqrt{2}} \cdot \frac{-4 - 3\sqrt{2}}{-4 - 3\sqrt{2}}$$

You may note that the numerator expands to $-62 - 43\sqrt{2}$ and the denominator simplifies to -2, so we can use the previous property to simply the expression to:

$$\frac{5 + 7\sqrt{2}}{-4 + 3\sqrt{2}} = (-62 - 43\sqrt{2}) \cdot \frac{1}{-2} = 31 + 21.5\sqrt{2}.$$

If $\alpha + \beta\sqrt{2}$ is such that α and β are rational numbers, then $(\alpha + \beta\sqrt{2})(\alpha - \beta\sqrt{2})$ will be a rational number $\alpha^2 - 2\beta^2$.

Definition: The "complex conjugate" of the complex number $z = \alpha + \beta j$, denoted by z^*, is the complex number $z^* = \alpha - \beta j$. Thus, the complex number of $-4 + 3j$ is $(-4 + 3j)^* = -4 - 3j$. One important property of the complex conjugate is that a complex number multiplied by its complex conjugate must be a non-negative real number:

$$zz^* = (\alpha + \beta j)(\alpha - \beta j) = \alpha^2 + \beta^2,$$

and $zz^* = 0$ if and only if $z = 0 + 0j$.

Like we did with radicals, we can do the same for complex numbers: given the two complex numbers $5 + 7j$ divided by $-4 + 3j$, by multiplying by the complex conjugate of the denominator over itself:

$$\frac{5 + 7j}{-4 + 3j} = \frac{5 + 7j}{-4 + 3j} \cdot 1 = \frac{5 + 7j}{-4 + 3j} \cdot \frac{-4 - 3j}{-4 - 3j}$$

You may note that the numerator expands to $1 - 43\sqrt{2}$ and the denominator simplifies to 25, so complex division is now reduced to complex multiplication and the division of a complex number by a positive real number, in this case, $(-4 + 3j)(-4 - 3j) = 4^2 + 3^2 = 25$:

$$\frac{5 + 7j}{-4 + 3j} = (1 - 43j) \cdot \frac{1}{25} = 0.04 - 1.72j.$$

You will observe that if $z = \alpha + \beta j$, then $zz^* = \alpha^2 + \beta^2$.

Thus,

$$\frac{\alpha + \beta j}{\gamma + \delta j} = \frac{(\alpha + \beta j)(\gamma - \delta j)}{\gamma^2 + \delta^2} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}j.$$

However, do not memorize this formula: just remember that $\frac{w}{z} = \frac{wz^*}{zz^*}$ and then work out the details.

Like real numbers, you can cancel terms out, so if w and z are complex numbers, then $\frac{w^3z^2}{wz^4} = \frac{w^2}{z^2} = \left(\frac{w}{z}\right)^2$. Similarly, like the real numbers, the only number you cannot divide by is $0 + 0j$: for every other complex number w, the ratio $\frac{z}{w}$ is well defined.

1.5 The inverse or reciprocal of a complex number

As a consequence, the reciprocal of z is

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*},$$

so for example, the reciprocal of $3 + 4j$ is $\frac{3-4j}{3^2+4^2} = \frac{3-4j}{25} = 0.12 - 0.16j$. You can multiply these together to get that $(3 + 4j)(0.12 - 0.16j) = 0.36 - 0.48j + 0.48j - 0.64j^2 = 1 + 0j$.

1.6 The length or absolute value of a complex number

The "length" or "absolute value" of a complex number $z = \alpha + \beta j$ is the distance to the origin if the point (α, β) was plotted on the Cartesian plane, or in other words, $|z| = \sqrt{\alpha^2 + \beta^2}$. For example, if $z = 4 - 3j$, then $|z| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$ and if $w = -3 + 1.25j$, then $|w| = \sqrt{(-3)^2 + 1.25^2} = \sqrt{10.5625} = 3.25$. You will notice that $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0 + 0j$.

Note that $|z|^2 = zz^*$ as $zz^* = \alpha^2 + \beta^2 = |z|^2$ and. Also, $\frac{w}{z} = \frac{wz^*}{|z|^2}$ and $\frac{1}{z} = \frac{z^*}{|z|^2}$.

Note that $|wz| = |w||z|$, just like the real numbers, and that $|z^{-1}| = \frac{1}{|z|}$ for a non-zero complex number z. The second is interesting, because on the left-hand side, you are calculating the complex reciprocal first, and only then calculating the absolute value. In the second, you calculate the absolute value of z first and then find the reciprocal of the result.

All complex numbers that have $|z| = 1$ form a circle in the complex plane, and this is called the "unit circle." All complex numbers such that $|z| \leq 1$ form what is called the "unit disk". All complex numbers such that $|z| < 1$ form what is called the "open unit disk," meaning that it does not include the boundary.

Just like two real numbers x and y are "close" to each other if $|x - y| < \epsilon$, two complex numbers w and z may be said to be "close" if $|w - z| < \epsilon$ where $|w - z|$ is the absolute value of the difference between the two complex numbers.

1.7 Properties of the complex conjugate

If w and z are complex numbers, then $z^{**} = z$, $(w + z)^* = w^* + z^*$ and $(wz)^* = w^* z^*$. Also, $z = z^*$ if and only if z is real, $z = -z^*$ if and only if z is imaginary, and $z^* = z^{-1}$ if and only if z lies on the unit circle.

1.8 Properties of complex numbers

Note that the complex numbers $0+0j$ and $1+0j$ have all the same properties as 0 and 1 do for real numbers:

1. $(0 + 0j)(\alpha + \beta j) = 0 + 0j$ for all complex numbers $z = \alpha + \beta j$.
2. $(1 + 0j)(\alpha + \beta j) = \alpha + \beta j$ for all complex numbers $z = \alpha + \beta j$.
3. For every complex number $z = \alpha + \beta j$, there is an "additive inverse" $-z = -\alpha - \beta j$ such that $z + (-z) = 0 + 0j$. The additive inverse may be found by multiplying a complex number by $-1 + 0j$.
4. For every non-zero complex numbers $z = \alpha + \beta j$, there is a "multiplicative inverse" $z^{-1} = \frac{\alpha - \beta j}{\alpha^2 + \beta^2}$ such that $zz^{-1} = 1 + 0j$.

Because of this, we often just write $0 + 0j$ as 0 and $1 + 0j$ as 1. In fact, we usually write $\alpha + 0j$ as α.

2 The square root and the quadratic formula

You are aware that for two non-negative numbers x and y, $\sqrt{xy} = \sqrt{x}\sqrt{y}$. Because we are now discussing $\sqrt{-1}$, we must emphasize that this is not true in general: For example, $1 = \sqrt{1} = \sqrt{(-1)(-1)}$, but if we claim that $\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$, then $1 = \sqrt{-1}\sqrt{-1} = j^2 = -1$, which is false. There is, however, one true statement: if z is any complex number and $x \geq 0$, then $\sqrt{xz} = \sqrt{x}\sqrt{z}$.

We can use this in finding the roots of the quadratic polynomial $x^2 + 8x + 41$:

$$\begin{aligned}\frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 41}}{2 \cdot 1} &= \frac{-8 \pm \sqrt{64 - 164}}{2} = \frac{-8 \pm \sqrt{-100}}{2} \\ &= \frac{-8 \pm \sqrt{100}\sqrt{-1}}{2} = \frac{-8 \pm 10j}{2} = -\frac{8}{2} \pm \frac{10j}{2} = -4 \pm 5j.\end{aligned}$$

3 Geometric series

You may recall that $1 + \gamma + \gamma^2 + \gamma^3 + \cdots = \sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}$ for real numbers so long as $|\gamma| < 1$. For example, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$. The same is true for complex numbers: If you calculate $1 + (0.4 + 0.2j) + (0.4 + 0.2j)^2 + \cdots$, you will find this converges to

$$\frac{1}{1 - (0.4 + 0.2j)} = \frac{1}{0.6 - 0.2j} = \frac{0.6 + 0.2j}{0.36 + 0.04} = \frac{0.6 + 0.2j}{0.4} = 1.5 + 0.5j.$$

You can try this out, as

$$\sum_{k=0}^{15} (0.4 + 0.2j)^k = 1.499999540084736 + 0.499995978498048j,$$

exactly and is clearly very close to $1.5 + 0.5j$.

4 Roots of a polynomial and the fundamental theorem of algebra

Recall that you learned polynomial division in secondary school. Thus, if you divided a non-constant polynomial $p(x)$ of degree n by $x - x_0$, this would result in a quotient polynomial $q(x)$ of degree $n-1$ plus the remainder equal to $p(x_0)$ so that $p(x) = (x - x_0)q(x) + p(x_0)$. For example,

$$x^5 - 3x^3 + 2x + 1 = (x+2)(x^4 - 2x^3 + x^2 - 2x + 6) - 11,$$

where $(-2)^5 - 3(-2)^3 + 2(-2) + 1 = -32 + 24 - 4 + 1 = -11$.

For example, if $p(x) = x^4 + 3x^3 - 9x^2 - 23x - 12$, then dividing this by $x - 2$ yields $q(x) = x^3 + 5x^2 + x - 21$ with a remainder of -54, and therefore $p(x) = q(x)(x-2) - 54$. If you expand $(x-2)(x^3+5x^2+x-21)-54$, you will indeed get back the original polynomial $p(x)$:

$$\begin{aligned}(x-2)(x^3+5x^2+x-21)-54 &= (x^4+5x^3+x^2-21x) - (2x^3+10x^2+2x-42) - 54\\ &= (x^4+3x^3-9x^2-23x+42) - 54\\ &= x^4+3x^3-9x^2-23x-12\end{aligned}$$

Therefore, a given real or complex number r and a polynomial $p(x)$, r is a root of the polynomial if and only if dividing the polynomial by $x - r$ has a remainder of zero. For example, given the quartic polynomial $p(x)$ defined above, we observe that $p(x) = (x-3)(x^3+6x^2+9x+4)+0$, $p(x) = (x+4)(x^3-x^2-5x-3)+0$ and $p(x) = (x+1)(x^3+2x^2-11x-12)+0$, and therefore 3, -4 and -1 are all roots of the polynomial $p(x)$.

A polynomial $p(x)$ has a "multiple root" at r if the quotient polynomial $q(x)$ also has a root at r:

1. A polynomial has a root at r of multiplicity equal to 1 if $p(r) = 0$, so $p(x) = (x-r)q(x)+0$ but $q(r) \neq 0$.
2. A polynomial has a root at r of multiplicity m if $p(x) = (x-r)q(x)+0$ (so $p(r) = 0$) and $q(x)$ has a root at r of multiplicity $m-1$.

For example, continuing with the example above:

- $p(x) = (x-3)(x^3+6x^2+9x+4)+0$ but $3^3 + 6\cdot 3^2 + 9\cdot 3 + 4 = 112$,
  so $p(x)$ has a root of multiplicity one at 3.
- $p(x) = (x+4)(x^3-x^2-5x-3)+0$ but $(-4)^3 - (-4)^2 - 5(-4) - 3 = -63$,
  so $p(x)$ has another root of multiplicity one at -4.
- $p(x) = (x+1)(x^3+2x^2-11x-12)+0$ and $x^3+2x^2-11x-12 = (x+1)(x^2+x-12)+0$ but $(-1)^2 + (-1) - 12 = -12$,
  so $p(x)$ has a root of multiplicity two at -1.

Theorem 1. If $p(x)$ is a polynomial with real coefficients of degree n, then $p(x)$ has **at most** n real roots if you count multiplicity.

For example, consider these cubic polynomials:

- The polynomial $x^3 - 12x$ has three roots of multiplicity 1 at $x = 0$ and $x = \pm 2\sqrt{3}$.
- The polynomial $x^3 - 12x + 16$ has a root of multiplicity 1 at $x = -4$ and a root of multiplicity 2 at $x = 2$.
- The polynomial $x^3 - 12x + 65$ has a root of multiplicity 1 at $x = -5$.

However, to give a few more example:

- The polynomial $x^2 - 4$ has two roots of multiplicity 1 at $x = \pm 2$.
- The polynomial x^2 has a root of multiplicity 2 at $x = 0$.
- The polynomial $x^2 + 1$ has no real roots.

Theorem 2. The fundamental theorem of algebra says that a polynomial of degree n with complex coefficients has exactly n complex roots if you count multiplicity.

As every real number r is the complex number $r + 0j$, it follows that every polynomial of degree n with real coefficients has exactly n complex roots (some of them possibly real) if you count multiplicity.

For example, the polynomial $x^3 + x^2 - 7x - 15$ has only one real root at $x = 3$, and therefore the two other roots must be non-real complex numbers.

There is a special property of the roots of a polynomial with real coefficients:

Theorem 3. If a polynomial $p(x)$ has real coefficients and z is a non-real root of $p(x)$ with multiplicity m, then so is z^*, the complex conjugate of z.

A complex number and its complex conjugate are said to constitute a "complex conjugate pair."

In the above example, because the polynomial $x^3 + x^2 - 7x - 15$ has real coefficients, its two complex roots must form a complex conjugate pair. Specifically,

$$x^3 + x^2 - 7x - 15 = (x - 3)(x + 2 - j)(x + 2 + j),$$

so the non-real complex roots are $-2 + j$ and $-2 - j$.

For example, the polynomial

$$\begin{aligned} p(x) = x^{12} &+ 5x^{11} - 27x^{10} + 11x^9 + 1000x^8 - 1338x^7 - 12890x^6 \\ &+ 8186x^5 + 27357x^4 + 20405x^3 + 97525x^2 + 10875x + 56250 \end{aligned}$$

must have twelve roots if you count multiplicity, and if any non-real complex number is a root, then so is its complex conjugate. In this case, this polynomial has a complex conjugate pair of roots of multiplicity two at $x = \pm j$, a complex conjugate pair of roots of multiplicity one at $x = 3 \pm 4j$, a real root of multiplicity one at $x = -2$, a real root of multiplicity two at $x = 3$ and a real root of multiplicity three at $x = -5$, for a total of twelve roots. Therefore,

$$p(x) = (x - j)^2(x + j)^2(x - 3 + 4j)(x - 3 - 4j)(x + 2)(x - 3)^2(x + 5)^3.$$

You are not expected to be able to know how to find all twelve roots, but we could approximate at least the roots by using Newton's method.

5 Newton's method

Given a polynomial $p(x)$ with at least one real root, you can find a real root using this algorithm:

1. Estimate a root with a value x_0.
2. Calculate $x_{k+1} \leftarrow x_k - \frac{p(x_k)}{p^{(1)}(x_k)}$ for $k = 0, 1, 2, 3, \ldots$ until $|x_{k+1} - x_k| < 10^{-12}$.
   Here, we use the $\leftarrow$ symbol to emphasize that we assign to x_{k+1} the result of the calculation on the right-hand side.
3. If this does not converge, the polynomial may not have a real root, so try again with a different x_0, but if you already have tried again, we are likely finished.
4. If it does converge, divide the polynomial by $p(x)$ by $x - xk + 1$ to get a polynomial $q(x)$ of one lower degree with the root we found removed:
   (a) If the resulting polynomial is of degree 0, we are done.
   (b) Otherwise, go back to Step 1 with the new polynomial $q(x)$.

For example, given the polynomial of degree 12 at the end of the last section, if we estimate a root with $x_0 = -3$, we iterate to get the sequence of points:

$$\begin{aligned} x_1 &= -2.2089249492900608519 \\ x_2 &= -2.0376884045188339208 \\ x_3 &= -2.0016434601040040205 \\ x_4 &= -2.0000033488809440001 \\ x_5 &= -2.0000000000139502783 \\ x_6 &= -2.0000000000000000000 \\ x_7 &= -2.0000000000000000000 \end{aligned}$$

Note that $|x_6 - x_5| = 1.39502783 \times 10^{-11}$, so we had to iterate one more time.

6 Euler's formula

Euler's formula relates the exponential function, the trigonometric functions and j through the formula

$$\mathrm{e}^{yj} = \cos(y) + \sin(y)j$$

If you substitute $y = \pi$ into this formula, you get

$$\mathrm{e}^{\pi j} = \cos(\pi) + \sin(\pi)j = -1 + 0 \cdot j = -1.$$

You may wonder how this is true, and for this, we turn to Taylor series:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots.$$

If we substitute $x = \beta j$, assuming β is real, into this formula, we get:

$$\mathrm{e}^{\beta j} = 1 + \beta j + \frac{(\beta j)^2}{2} + \frac{(\beta j)^3}{6} + \frac{(\beta j)^4}{24} + \frac{(\beta j)^5}{120} + \frac{(\beta j)^6}{720} + \frac{(\beta j)^7}{5040} + \cdots .$$

Now, $(\beta j)^n = \beta^n j^n$:

$$\mathrm{e}^{\beta j} = 1 + \beta j + \frac{\beta^2 j^2}{2} + \frac{\beta^3 j^3}{6} + \frac{\beta^4 j^4}{24} + \frac{\beta^5 j^5}{120} + \frac{\beta^6 j^6}{720} + \frac{\beta^7 j^7}{5040} + \cdots .$$

However, $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$, and so on, so we may simplify this:

$$\begin{aligned}
\mathrm{e}^{\beta j} &= 1 + \beta j + \frac{\beta^2 \cdot (-1)}{2} + \frac{\beta^3 \cdot (-j)}{6} + \frac{\beta^4 \cdot 1}{24} + \frac{\beta^5 \cdot j}{120} + \frac{\beta^6 \cdot (-1)}{720} + \frac{\beta^7 \cdot (-j)}{5040} + \cdots \\
&= 1 + \beta j - \frac{\beta^2}{2} - \frac{\beta^3}{6} j + \frac{\beta^4}{24} + \frac{\beta^5}{120} j - \frac{\beta^6}{720} - \frac{\beta^7}{5040} j + \cdots
\end{aligned}$$

We observe that terms with even powers of β involve only real numbers, while terms with odd powers of β are real numbers multiplied by j. This allows us to separate the sum into two parts:

$$\mathrm{e}^{\beta j} = \left(1 - \frac{\beta^2}{2} + \frac{\beta^4}{24} - \frac{\beta^6}{720} + \cdots\right) + \left(\beta - \frac{\beta^3}{6} + \frac{\beta^5}{120} - \frac{\beta^7}{5040} + \cdots\right) j$$

You will notice that the first is the Taylor series for cosine, and the second is for sine, and thus, we have:

$$\mathrm{e}^{\beta j} = \cos(\beta) + \sin(\beta) j$$

You should note that $\mathrm{e}^{\beta j}$ is a point on the unit circle with an angle of β radians relative to the real axis.

Now, because $\mathrm{e}^{x+y} = \mathrm{e}^x \mathrm{e}^y$, it follows that:

$$\mathrm{e}^{\alpha + \beta j} = \mathrm{e}^{\alpha} \mathrm{e}^{\beta j} = \mathrm{e}^{\alpha} \left(\cos(\beta) + \sin(\beta) j\right).$$

You will note that if $\beta = 0$ in this formula, this simplifies to $\mathrm{e}^{\alpha}(1 + 0j) = \mathrm{e}^{\alpha}$.

To give a simple example, note that if $z = 0.1 - 0.2j$, the claim is that

$$\begin{aligned}
\mathrm{e}^{0.1 - 0.2j} &= \mathrm{e}^{0.1}(\cos(-0.2) + \sin(-0.2) j) \\
&= 1.105170918075648 \cdot (0.980066577841242 - 0.198669330795061 j) \\
&= 1.083141079608063 - 0.219563566708252 i
\end{aligned}$$

While it is a little tedious, you could calculate:

$$\begin{aligned}
&1 + (0.1 - 0.2j) + \frac{(0.1 - 0.2j)^2}{2} + \frac{(0.1 - 0.2j)^3}{6} + \frac{(0.1 - 0.2j)^4}{24} + \frac{(0.1 - 0.2j)^5}{120} + \frac{(0.1 - 0.2j)^6}{720} \\
&\quad = 1.0831410791\overline{6} - 0.21956356\overline{1} j,
\end{aligned}$$

which seems to be converging to the calculation above.

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