Instructions

- You may rip off the last three pages as soon as you sit down.
- There are 62 marks available. It will be marked out of 60.
- No aides.
- Turn off all electronic media and store them under your desk.
- You may ask only one question during the examination: "May I go to the washroom?"
- Asking any other question will result in a deduction of 5 marks from the exam grade.
- If you think a question is ambiguous, write down your assumptions and continue.
- Do not leave during first hour or after there are only 15 minutes left.
- Do not stand up until all exams have been picked up.
- There are questions on both sides of the pages.
- If a question only asks for an answer, you do not have to show your work to get full marks; however, if your answer is wrong and no rough work is presented to show your steps, no part marks will be awarded.
- Answer the questions in the spaces provided. If you require additional space to answer a question, please use the provided blank page and refer to this page in your solutions.

1. (2 points) What is the value stored in this floating-point number in decimal recalling that 0.1 in binary is 0.5 in decimal, 0.01 is 0.25, 0.001 is 0.125, etc.

```
c025000000000000    1 10000000010 0101000000...0
```

2. (3 points) Find the coefficient c of the ch^2 term in the approximation of the first derivative

$$y^{(1)}(t) \approx \frac{3y(t) - 4y(t-h) + y(t-2h)}{2h} + ch^2$$

by taking appropriate linear combinations of

$$y(t-h) \approx y(t) - y^{(1)}(t)h + \frac{1}{2}y^{(2)}(t)h^2 - \frac{1}{6}y^{(3)}(t)h^3$$

and the 3rd-order Taylor polynomial for $y(t-2h)$ and then isolating $y^{(1)}(t)$.

3. (4 points) Find the error of Newton's method starting with the observation that if x_k is an approximation of the root, then

$$f(x) = f(x_k) + f^{(1)}(x_k)(x - x_k) + \frac{1}{2} f^{(2)}(\xi)(x - x_k)^2$$

where ξ is between x and x_k, so this must also be true if x is a root. If x is a root, then the error of x_k is $x - x_k$. Show that the error of the next approximation is a coefficient times the error of the current approximation squared.

4. (3 points) We have the following approximation of the second derivative:

$$y^{(2)}(t) \approx \frac{y(t) - 2y(t-h) + y(t-2h)}{h^2}.$$

However, we also know that $\frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} y(t)\right)$ and that $y^{(1)}(t) \approx \frac{y(t)-y(t-h)}{h}$.

Show that by applying the first derivative formula to the first derivative formula leads to the approximation found in the second derivative formula.

Next, argue why this shows that the second derivative formula must be $O(h)$.

5. (2 points) Given the initial-value problem $y^{(1)}(t) = -y(t) + t + 1$ with $y(1) = 2$, apply one step of Heun's method to approximate $y(1.5)$.

6. (4 points) Suppose you are applying the Dormand-Prince method approximating a solution to the initial-value problem $y^{(1)}(t) = 3\sin(y(t))y(t)t + 1$ with $y(23) = 31$, and you are approximating the next value with $h = 0.1$. Suppose that your error per unit time is $\epsilon_{\text{abs}} = 0.01$ and you get two approximations: $y = 34$ and $z = 29$. Indicate if you accept or reject $z = 29$ as the approximation of $y(23.1)$. Then:

    1. If you accept $z = 29$, then what would the value of h be with the next step, and which t-value would you be approximating next?
    2. If you reject $z = 29$, then what would the value of h be with the next step, and which t-value would you be approximating next.

7. (3 points) Write down the system of linear equations that must be solved in order to find the best-fitting line passing through the three points $(0, 5)$, $(1, 7)$ and $(3, 6)$. You do not have to solve this, but it must be a system that has at least one solution.

8. (4 points) Approximate $x(0.2)$ using Euler's method by first converting this second-order initial-value problem into a system of first-order initial-value problems, and then applying one step of Euler's method. Indicate which entry approximates $x(0.2)$ and which approximates $x^{(1)}(0.2)$, and next, what is an approximation of $x^{(2)}(0.2)$?

$$\begin{aligned} x^{(2)}(t) &= -x(t) + x^{(1)}(t) + 1 \\ x(0) &= 18 \\ x^{(1)}(0) &= 12 \end{aligned}$$

What method would you use to approximate a value of the solution between 0 and 0.2?

9. (3 points) Suppose you have applied the techniques in class to find a least-squares best-fitting quadratic through the last 11 points, each measuring electric current, and you get that the least-squares best-fitting quadratic is $1.50 + 0.36s + 0.18s^2$, as described in class. If these readings are being taken once per one hundred seconds, what is the best approximation of the charge passing through that point in the most recent time step?

Remember, in class, we shifted and scaled so that the most recent reading is at 0, the previous reading is at -1, etc.

10. (4 points) Suppose we have an initial state of the temperature along a metal bar at time $t = 0$ with $u_{\text{init}}(x) = 0$ if $x \leq 0.45$ and $u_{\text{init}}(x) = 1$ if $x > 0.45$ where $0 \leq x \leq 1$. The boundary conditions are $u_a(t) = 0$ and $u_b(t) = 1$. We will use an $h = 0.1$. If $\Delta t = 0.01$ and $\alpha = 0.2$, first approximate the state at each of the 9 interior points at time $t = 0.01$ and demonstrate that they are as shown below. Instead of explicit calculations, you can explain why many of the points are unchanged in their values. Next, **proceed to approximate the solutions at** $t = 0.02$:

x :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$t = 0$	0	0	0	0	0	1	1	1	1	1	1
$t = 0.01$	0	0	0	0	0.2	0.8	1	1	1	1	1
$t = 0.02$	0	?	?	?	?	?	?	?	?	?	1

11. (3 points) Suppose you have a system that satisfies Laplace's equation (so it is in steady state) where you have the following approximation:

$*$	$*$	$*$	$*$	$*$	10
$*$	u_1	u_2	u_3	u_4	10
$*$	u_5	u_6	26	u_7	10
$*$	4	4	4	4	10

where the stars indicate boundary points that are insulated. Write down the system of linear equations in the form of an augmented matrix that must be solved to approximate the solutions at the points u_1 through u_7.

12. (3 points) Suppose you have a second-order linear boundary-value problem with $a = 0$ and $b = 1$ where the right-hand boundary (at $b = 1$) is insulated and you determined that with $n = 10$ that $u(0.8) \approx u_8 = 1.9$ and $u(0.9) \approx u_9 = 2.2$. Which of the formulas approximating the derivative $\frac{u_{10}-u_9}{0.1}$ and $\frac{3u_{10}-4u_9+u_8}{0.2}$ would you use to calculate u_{10} to approximate $u(1)$, why, and what is that the better approximation of u_{10} under the assumption it is an insulated boundary (so the derivative at that point is zero).

13. (3 points) The advection-diffusion equation in one dimension is

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is the quantity, v is the advection velocity (speed of flow), and α is the diffusion coefficient. Without the velocity term, we have the heat equation, where we found

$$u_{k,\ell+1} = u_{k,\ell} + \alpha\Delta t\frac{u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}}{h^2}.$$

How does this change if we convert the advection-diffusion equation into a finite-difference equation using the appropriate $O(h^2)$ approximation of the first partial derivative of u with respect to x? You can just write down the answer, or you can derive it.

If you just write down the correct answer with no derivation, you will receive full marks; however, a more detailed derivation may result in part marks if the solution is wrong.

14. (2 points) In finding the inverse of a matrix, which would have the least numerical error, and which would have the most numerical error:

    1. A diagonal matrix with all non-zero entries on the diagonal.
    2. An orthogonal matrix where all columns have a 2-norm of 1 and are mutually orthogonal.
    3. A full matrix where the determinant is non-zero.

    Minus 1 mark for each entry that is out of order (with a minimum of 0).

15. (3 points) If $\mathbf{v}_1$ through $\mathbf{v}_4$ are four vectors, then the Bezier curve

$$(1-t)^3\mathbf{v}_1 + 3(1-t)^2 t\mathbf{v}_2 + 3(1-t)t^2\mathbf{v}_3 + t^3\mathbf{v}_4$$

    is a linear combination of these four vectors where t takes on a value between 0 and 1. Note that when $t = 0$, this equals $\mathbf{v}_1$ and when $t = 1$, this equals $\mathbf{v}_4$. Show that for any value of $0 \leq t \leq 1$, this is a convex combination of the four vectors. You may recall that

    1. $(1-t)^3 = 1 - 3t + 3t^2 - t^3$,
    2. $(1-t)^2 t = t - 2t^2 + t^3$ and
    3. $(1-t)t^2 = t^2 - t^3$.

    Also, you may note that $t \geq 0$ and $1 - t \geq 0$ for $0 \leq t \leq 1$.

16. (4 points) Approximate the integral $\int_0^2 x^2 \mathrm{d}x$ using one step of the trapezoidal rule, one step of Simpson's rule, and one step of the centered four-point rule. Should your answer be the same for the second and third approximations? Why or why not?

17. (2 points) Show that if $\phi = \frac{1+\sqrt{5}}{2}$, then $\phi^{-1} = \frac{\sqrt{5}-1}{2}$. Next, argue that if the width of the initial interval is $b_0 - a_0$, then after n steps of the golden ration search, the width of the interval will be reduced to $b_n - a_n = (b_0 - a_0)/\phi^n$.

18. (4 points) Using our four decimal-digit representation of floating-point numbers, we always had to round the result of any calculation to four digits before proceeding. Using this, approximate the smaller root of the polynomial $0.003x^2+5x+0.01$ using both formulas $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-2c}{b+\sqrt{b^2-4ac}}$ and explain why one answer is better. The correct answer is approximately -0.0020000024000057600.

19. (2 points) What does Horner's rule minimize when evaluating a polynomial at a point versus other methods of evaluating a polynomial at a point?

    By requiring that $|x| < 1$ before applying Horner's rule (which is why we always shifted and scaled), what other potential problem do we try to avoid? Hint: see the previous question.

20. (4 points) Apply one step of gradient descent to find the minimum of $f(x, y) = x^2 + y^2 + x - 2y + 0.5xy + 1$ starting with the initial point $x_0 = y_0 = 0$. Note that $\mathbf{0} - s(\vec{\nabla} f)(\mathbf{0})$ simplifies to $-s(\vec{\nabla} f)(\mathbf{0})$, which can be easily substituted into the function f, where one can quickly find the minimum of the resulting quadratic to subsequently find x_1 and y_1. Note that $\frac{5}{8} = 0.625$ and twice this is 1.25.

USE THIS PAGE IF ADDITIONAL SPACE IS REQUIRED
Clearly state the question number being answered and refer the marker to this page.

YOU MAY RIP THESE LAST THREE PAGES OFF

Floating-point representations: $\pm$`EENMMM` represents $\pm$`N.MMM`$\times 10^{\texttt{EE}-49}$ and the 64 bits `seeeeeeeeeeebbbbbb`$\cdots$ represents

$$(-1)^{\texttt{s}}\texttt{1.bbbbbb}\cdots\texttt{b} \times 2^{\texttt{eeeeeeeeeee}-01111111111}$$

where `0b01111111111` $= 1023 =$ `0x3ff`. Recall 1 is `+491000` or `0x3ff0000000000000`.

0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

Given n real or complex numbers or vectors $x_1, \ldots, x_n$ and n real or complex numbers $w_1, \ldots, w_n$, then $\sum_{k=1}^{n} w_k x_k$ is:

1. a linear combination of the x-values if there are no restrictions on the weights,
2. a weighted average if $\sum_{k=1}^{n} w_k = 1$, and
3. a convex combination if the weights form a weighted average and each $w_k \geq 0$.

Fixed-point theorem: To approximate a solution to $x = f(x)$, choose x_0 and let $x_k \leftarrow f(x_{k-1})$.
Gaussian elimination with partial pivoting: This is the Gaussian elimination algorithm but always swapping appropriate rows so that the largest entry in absolute value is in the pivot position (the row that will be used to eliminate entries in that column in subsequent rows).
***n*th-order Taylor series:** If h is small, expanding around x yields:

$$f(x+h) = \left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k \right) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) h^{n+1}$$

where $x \leq \xi \leq x + h$. Otherwise, if x is close to x_0, expanding around x_0 yields:

$$f(x) = \left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0) (x - x_0)^k \right) + \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

where $x_0 \leq \xi \leq x$.

The examples of binary search and interpolation search are not required for this course: they are provided as examples of different bracketing algorithms.

```
double horner( double       const a[],
               unsigned int const degree,
               double       const x ) {
    // The coefficient of xˆk is a[k]
    double result{ a[degree] };

    for ( std::size_t k{degree - 1}; k < degree; --k ) {
        result = result*x + a[k];
    }

    return result;
}
```

Noise: Averaging noisy values with zero bias mitigates the effect, while differentiating noisy values magnifies the effect. Use interpolating polynomials if the data is accurate and precise, but use least squares best-fitting polynomials if the data is accurate but not precise (that is, the data has significant noise). If the data is not accurate, we cannot recover the underlying signal.
Evaluating interpolating polynomials: For interpolating between t_k and t_{k-1} where t_k is the time of the most recent data point, shift and scale to $\ldots, -2.5, -1.5, -0.5$ and 0.5 to ensure that $-0.5 < \delta < 0.5$ to evaluate the polynomial at the point $\frac{t_{k-1}+t_k}{2} + \delta h$ where h is the time step between readings. Note, you do not have to know these formulas explicitly; rather, you must understand the idea behind deriving these. For example, why to we shift and scale so that our choice of δ is such that $|\delta| < 0.5$.

Derivatives:
Centered three-point:

$$f^{(1)}(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6} f^{(3)}(\xi) h^2$$

Backward two-point:

$$y^{(1)}(t) = \frac{y(t) - y(t-h)}{h} + \frac{1}{2} y^{(2)}(\tau) h$$

Backward three-point:

$$y^{(1)}(t) = \frac{3y(t) - 4y(t-h) + y(t-2h)}{2h} + \frac{1}{3} y^{(3)}(t) h^2 + \mathrm{O}\left(h^3\right)$$

Second derivatives:
Centered three-point:

$$f^{(2)}(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{1}{12} f^{(4)}(\xi) h^2$$

Backward three-point:

$$y^{(2)}(t) = \frac{y(t) - 2y(t-h) + y(t-2h)}{h^2} + y^{(3)}(\tau) h$$

Backward four-point:

$$y^{(2)}(t) = \frac{2y(t) - 5y(t-h) + 4y(t-2h) - y(t-3h)}{h^2} + \frac{11}{12} y^{(4)}(t) h^2 + \mathrm{O}\left(h^3\right)$$

Integrals:
Two-point (trapezoidal rule):

$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \left(\frac{1}{2}f(x_{k-1}) + \frac{1}{2}f(x_k)\right)h - \frac{1}{12}f^{(2)}(\xi)\,h^3$$

Centered four-point:

$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \left(-\frac{1}{24}f(x_{k-2}) + \frac{13}{24}f(x_{k-1}) + \frac{13}{24}f(x_k) - \frac{1}{24}f(x_{k+1})\right)h - \frac{11}{720}f^{(4)}(t_k)\,h^5 + \mathrm{O}\left(h^6\right)$$

Simpson's rule:

$$\int_{x_{k-1}}^{x_{k+1}} f(x)\,\mathrm{d}x = \left(\frac{1}{6}f(x_{k-1}) + \frac{4}{6}f(x_k) + \frac{1}{6}f(x_{k+1})\right)(2h) - \frac{1}{90}f^{(4)}(\xi)\,h^5$$

Backward three-point (half Simpson's rule):

$$\int_{t_{k-1}}^{t_k} y(t)\,\mathrm{d}x = \left(\frac{5}{12}y(t_k) + \frac{8}{12}y(t_{k-1}) - \frac{1}{12}y(t_{k-2})\right)h - \frac{1}{24}y^{(3)}(t_k)\,h^4 + \mathrm{O}\left(h^5\right)$$

Backward four-point:

$$\int_{t_{k-1}}^{t_k} y(t)\,\mathrm{d}x = \left(\frac{9}{24}y(t_k) + \frac{19}{24}y(t_{k-1}) - \frac{5}{24}y(t_{k-2}) + \frac{1}{24}y(t_{k-3})\right)h + \frac{19}{720}y^{(4)}(t_k)\,h^5 + \mathrm{O}\left(h^6\right)$$

As Simpson's rule spans two time intervals, it is less useful, but it is interesting with its comparison with the trapezoidal rule applied twice versus one application of Simpson's rule. It also corresponds with the 4th-order Runge Kutta method.
Any integral formula can be applied repeatedly on the interval $[a, b]$ by dividing the interval into n equally-spaced sub-intervals of width $h = \frac{b-a}{n}$ and then setting $x_k = a + kh$ or $t_k = a + kh$.

Least squares: In general, if we want to find the best approximation of an n-dimensional vector $\mathbf{y}$ by a linear combination of m vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ (where $m < n$), we create the matrix $V = (\mathbf{v}_1 \cdots \mathbf{v}_m)$ and solve $V^\top V\boldsymbol{\alpha} = V^\top \mathbf{y}$. More specific to this course, having shifted and scaled the n most recent t-values onto $0, -1, -2, \ldots, -n+1$, with y values $\mathbf{y} = (y_k, y_{k-1}, y_{0-2}, \ldots, y_{k-n+1})$, we solve $V^\top V\boldsymbol{\alpha} = V^\top \mathbf{y}$ for the coefficients of the least-squares best-fitting polynomial, generally of degree one (linear or $\alpha_1 t + \alpha_0$) or two (quadratic or $\alpha_2 t^2 + \alpha_1 t + \alpha_0$). We can find the $2\times n$ or $3\times n$ matrix to calculate $\boldsymbol{\alpha} = \left(V^\top V\right)^{-1} V^{\mathrm{T}} \mathbf{y}$.

Value being estimated	Linear estimation
$y(t_k)$	α_0
$y(t_k + h)$	$\alpha_0 + \alpha_1$
$y^{(1)}(t_k)$	α_1/h
$\int_{t_k-h}^{t_k} y(\tau)\mathrm{d}\tau$	$(\alpha_0 - \alpha_1/2)h$
$\int_{t_k}^{t_k+h} y(\tau)\mathrm{d}\tau$	$(\alpha_0 + \alpha_1/2)h$

Value being estimated	Quadratic estimation
$y(t_k)$	α_0
$y(t_k + h)$	$\alpha_0 + \alpha_1 + \alpha_2$
$y^{(1)}(t_k)$	α_1/h
$y^{(2)}(t_k)$	$2\alpha_2/h^2$
$\int_{t_k-h}^{t_k} y(\tau)\mathrm{d}\tau$	$(\alpha_0 - \alpha_1/2 + \alpha_2/3)h$
$\int_{t_k}^{t_k+h} y(\tau)\mathrm{d}\tau$	$(\alpha_0 + \alpha_1/2 + \alpha_2/3)h$

Root finding:

- Bisection: Let $m_k \leftarrow \frac{a_k+b_k}{2}$ and update that endpoint that has the value of the function have the same sign as $f(m_k)$.
- Newton's method: $x_{k+1} \leftarrow x_k - \frac{f(x_k)}{f^{(1)}(x_k)}$.
- Secant method: $x_{k+1} \leftarrow x_k - \frac{f(x_k)}{\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}} = x_k - \frac{f(x_k)(x_k-x_{k-1})}{f(x_k)-f(x_{k-1})}$.
- Inverse quadratic interpolation: Find the constant coefficient of the polynomial interpolating (y_{k-2}, x_{k-2}), (y_{k-1}, x_{k-1}) and (y_k, x_k).
- Newton's method in n dimensions: Given the approximation $\mathbf{u}_k$ to $\mathbf{f}(\mathbf{u}) = \mathbf{0}$, solve $J(\mathbf{f})(\mathbf{u}_k)\Delta\mathbf{u}_k = -\mathbf{f}(\mathbf{u}_k)$ and then let $\mathbf{u}_{k+1} \leftarrow \mathbf{u}_k + \Delta\mathbf{u}_k$.

Initial-value problems (IVPs): Given the ordinary-differential equation (ODE) and initial value

$$y^{(1)}(t) = f(t, y(t)) \text{ and } y(t_0) = y_0,$$

we will approximate $y_{k+1} \approx y(t_{k+1})$.

Given the system of ODEs and initial values

$$\mathbf{y}^{(1)}(t) = \mathbf{f}(t, \mathbf{y}(t)) \text{ and } \mathbf{y}(t_0) = \mathbf{y}_0$$

we will approximate $\mathbf{y}_{k+1} \approx \mathbf{y}(t_{k+1})$.

Euler's method:

$$y_{k+1} \leftarrow y_k + hf(t_k, y_k) \quad \mathbf{y}_{k+1} \leftarrow \mathbf{y}_k + h\mathbf{f}(t_k, \mathbf{y}_k)$$

Heun's method:

$$\begin{array}{ll} s_0 \leftarrow f(t_k, y_k) & \mathbf{s}_0 \leftarrow \mathbf{f}(t_k, y_k) \\ s_1 \leftarrow f(t_k + h, y_k + hs_0) & \mathbf{s}_1 \leftarrow \mathbf{f}(t_k + h, \mathbf{y}_k + h\mathbf{s}_0) \\ y_{k+1} \leftarrow y_k + h\frac{s_0+s_1}{2} & \mathbf{y}_{k+1} \leftarrow \mathbf{y}_k + h\frac{\mathbf{s}_0+\mathbf{s}_1}{2} \end{array}$$

The 4^{th}-order Runge-Kutta method:

$$\begin{array}{ll} s_0 \leftarrow f(t_k, y_k) & \mathbf{s}_0 \leftarrow \mathbf{f}(t_k, y_k) \\ s_1 \leftarrow f\left(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hs_0\right) & \mathbf{s}_1 \leftarrow \mathbf{f}\left(t_k + \frac{1}{2}h, \mathbf{y}_k + \frac{1}{2}h\mathbf{s}_0\right) \\ s_2 \leftarrow f\left(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hs_1\right) & \mathbf{s}_2 \leftarrow \mathbf{f}\left(t_k + \frac{1}{2}h, \mathbf{y}_k + \frac{1}{2}h\mathbf{s}_1\right) \\ s_3 \leftarrow f\left(t_k + h, y_k + hs_2\right) & \mathbf{s}_3 \leftarrow \mathbf{f}\left(t_k + h, \mathbf{y}_k + h\mathbf{s}_2\right) \\ y_{k+1} \leftarrow y_k + h\frac{s_0+2s_1+2s_2+s_3}{6} & \mathbf{y}_{k+1} \leftarrow \mathbf{y}_k + h\frac{\mathbf{s}_0+2\mathbf{s}_1+2\mathbf{s}_2+\mathbf{s}_3}{6} \end{array}$$

A single step of these three methods are $O(h^2)$, $O(h^3)$ and $O(h^5)$, respectively; however, multiple steps are one order less: $O(h)$, $O(h^2)$ and $O(h^4)$, respectively.

For Euler's method, the error of one step may be found from Taylor series:

$$y(t+h) = y(t) + y^{(1)}(t)h + \frac{1}{2}y^2(\tau)h^2$$

where

$$y(t_{k+1}) = y_k + f(t_k, y_k)h + \frac{1}{2}y^2(\tau)h^2$$

assuming that y_k is exact.

The adaptive Euler-Heun method Given an ODE and an initial value together with a maximum absolute error per unit step ϵ_{abs}, start with an initial step size h and determine both minimum and maximum step sizes h_{min} and h_{max}, respectively. Also, let $k \leftarrow 0$.

1. If $h < h_{min}$, set $h \leftarrow h_{min}$, and if $h > h_{max}$, set $h \leftarrow h_{max}$.
2. Given y_k, calculate $s_0 \leftarrow f(t_k, y_k)$ and $s_1 \leftarrow f(t_k + h, y_k + hs_0)$.
3. Estimate $y(t_k + h)$ with
   - $y \leftarrow y_k + hs_0$ (the worse approximation), and
   - $z \leftarrow y_k + h\frac{s_0+s_1}{2}$ (the better approximation).
4. Let $a \leftarrow \frac{h\epsilon_{\text{abs}}}{2|y-z|}$, and
   - if $a \geq 1$ or $h = h_{min}$, let $t_{k+1} \leftarrow t_k + h$ and let $y_{k+1} \leftarrow z$ and then set $h \leftarrow 0.9ah$ and $k \leftarrow k + 1$, and we will continue with the next step;
   - otherwise $a < 1$ and we will try again.
5. If $0.9a \leq 0.5$, set $h \leftarrow 0.5h$ (don't shrink h by more than a factor of two),
   else if $0.9a \geq 2$, set $h \leftarrow 2h$ (don't grow h by more than a factor of two),
   else set $h \leftarrow 0.9ah$.

Given a system of ODEs and initial values with a similar set-up:

1. If $h < h_{min}$, set $h \leftarrow h_{min}$, and if $h > h_{max}$, set $h \leftarrow h_{max}$.
2. Given y_k, calculate $\mathbf{s}_0 \leftarrow \mathbf{f}(t_k, \mathbf{y}_k)$ and $\mathbf{s}_1 \leftarrow \mathbf{f}(t_k + h, \mathbf{y}_k + h\mathbf{s}_0)$.
3. Estimate $\mathbf{y}(t_k + h)$ with
   - $\mathbf{y} \leftarrow \mathbf{y}_k + h\mathbf{s}_0$ (the worse approximation), and
   - $\mathbf{z} \leftarrow \mathbf{y}_k + h\frac{\mathbf{s}_0+\mathbf{s}_1}{2}$ (the better approximation).

4. Let $a \to \frac{h\epsilon_{\text{abs}}}{2\|\mathbf{y}-\mathbf{z}\|_2}$ where $\|\cdot\|_2$ is the 2-norm (or Euclidean norm), and
   - if $a \geq 1$ or $h = h_{min}$, let $t_{k+1} \leftarrow t_k + h$ and let $\mathbf{y}_{k+1} \leftarrow \mathbf{z}$ and then set $h \leftarrow 0.9ah$ and $k \leftarrow k+1$, and we will continue with the next step;
   - otherwise $a < 1$ and we will try again.
5. If $0.9a \leq 0.5$, set $h \leftarrow 0.5h$ (don't shrink h by more than a factor of two), else if $0.9a \geq 2$, set $h \leftarrow 2h$ (don't grow h by more than a factor of two), else set $h \leftarrow 0.9ah$.

Boundary-value problems (BVPs) Given a 2nd-order ODE $u^{(2)}(x) = f(x, u(x), u^{(1)}(x))$ with two boundary conditions $u(a) = u_a$ and $u(b) = u_b$, we can approximate a solution to this BVP as follows. First, create the IVP $u^{(1)}(x) = f(x, u(x), u^{(1)}(x))$ with the two initial conditions $u(a) = u_a$ and $u^{(1)}(a) = s$ where s is an initial slope we can choose. Let us use any technique you wish (preferably Dormand Prince), and let $u_s(x)$ be an approximation to the solution of this IVP with the initial slope s. Then the approximation of $u(b)$ using the this technique and initial slope is $u_s(b)$. Proceed as follows:

1. Let $s_0 = \frac{u_b - u_a}{b-a}$ and find $u_{s_0}(b)$. If $u_{s_0}(b) = u_b$, we are done, otherwise, continue.
2. Let $s_1 = \frac{2u_b - u_{s_0}(b) - u_a}{b-a}$ and find $u_{s_1}(b)$. If $u_{s_1}(b) = u_b$, we are done, otherwise, continue.
3. Define the function $f(s) = u_b - u_s(b)$, which is a function of a single variable s, and use s_0 and s_1 as the first two approximations for the secant method.

Linear boundary-value problems (BVPs) Given a 2nd-order linear ODE (LODE) $c_2(x)u^{(2)}(x) + c_1(x)u^{(1)}(x) + c_0(x)u(x) = g(x)$ with two boundary conditions $u(a) = u_a$ and $u(b) = u_b$, we convert the LODE into a linear finite-difference equation and let $x_k = a + hk$ for $h = \frac{b-a}{n}$, so if we define for each $k = 1, ..., n-1$ the three values $p_k = 2c_2(x_k) - hc_1(x_k)$, $q_k = -4c_2(x_k) + 2h^2c_0(x_k)$ and $r_k = 2c_2(x_k) + hc_1(x_k)$, we have

$$p_k u_{k-1} + q_k u_k + r_k u_{k+1} = 2h^2 g(x_k)$$

For Dirichlet boundary conditions, the corresponding linear equations are:

$$q_1 u_1 + r_1 u_2 = 2h^2 g(x_1) - p_1 u_a$$

$$p_{n-1} u_{n-2} + q_{n-1} u_{n-1} = 2h^2 g(x_{n-1}) - r_{n-1} u_b$$

For Neumann boundary conditions, the corresponding linear equations are:

$$\left(q_1 + \frac{4}{3}p_1\right) u_1 + \left(r_1 - \frac{1}{3}p_1\right) u_2 = 2h^2 g(x_1) + \frac{2}{3}hp_1 u_a^{(1)}$$

$$\left(p_{n-1} - \frac{1}{3}r_{n-1}\right) u_{n-2} + \left(q_{n-1} + \frac{4}{3}r_{n-1}\right) u_{n-1} = 2h^2 g(x_{n-1}) - \frac{2}{3}hr_{n-1}u_b^{(1)}$$

after which

$$u_0 \leftarrow -\frac{2}{3}hu_a^{(1)} + \frac{4}{3}u_1 - \frac{1}{3}u_2$$

$$u_n \leftarrow \frac{2}{3}hu_b^{(1)} + \frac{4}{3}u_{n-1} - \frac{1}{3}u_{n-2}$$

and this simplifies for insulated boundary conditions where $u_a^{(1)} = 0$ or $u_b^{(1)} = 0$.

Heat equation: For the heat equation $\frac{\partial u}{\partial t} = \alpha \nabla^2 u$, we have $u_{init}(x)$, $u_a(t)$ and $u_b(t)$, and we convert the partial-differential equation (PDE) into a finite-difference equation with $x_k \leftarrow a + kh$ and $t_\ell \leftarrow t_0 + \ell\Delta t$ so that $u_{k,0} \leftarrow u_{init}(x_k)$,

$$u_{k,\ell+1} \leftarrow u_{k,\ell} + \frac{\alpha\Delta t}{h^2}\left(u_{k-1,\ell} - 2u_{k,\ell} + u_{k+1,\ell}\right)$$

where $u_{0,\ell} \leftarrow u_a(t_\ell)$ and $u_{n,\ell} \leftarrow u_b(t_\ell)$.

Laplace's equation: On a grid $x_j = a_x + jh$ and $y_k \leftarrow a_y + kh$, we convert the PDE into the linear finite difference equation:

$$4u_{j,k} - u_{j-1,k} - u_{j+1,k} - u_{j,k-1} - u_{j,k+1} = 0$$

replacing any points on the boundary with their boundary value. Note that each value is the average of the surrounding points. If there are insulated boundaries, each entry must be the average of all the points around it that are not insulated boundary points.

Newton's: Use Newton's method on $f^{(1)}(x)$.
Golden ratio search: To minimize, given an interval $[a, b]$ with a minimum on that interval, let $m_1 \leftarrow b-(b-a)/\phi$ and $m_2 \leftarrow a+(b-a)/\phi$ and update $a \leftarrow m_1$ if $f(m_2) < f(m_1)$ and update $b \leftarrow m_2$ otherwise.
Successive parabolic interpolation: Formula not necessary: given three points, find the interpolating polynomial $ax^2 + bx + c$ and let the next point be $-\frac{b}{2a}$.
Newton's: Use Newton's method on $\vec{\nabla} f$.
Gradient descent: Calculate or approximate $\vec{\nabla} f(\mathbf{x}_k)$, and then use a one-dimensional algorithm to find a minimum of $f(\mathbf{x}_k - s\vec{\nabla} f(\mathbf{x}_k))$ and when we find the value s_k that gives us the minimum, set $\mathbf{x}_{k+1} \leftarrow \mathbf{x}_k - s_k \vec{\nabla} f(\mathbf{x}_k)$.