Buffer overflow bounds for multiplexed regulated traffic streams

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Abstract

In this paper, we present performance bounds for multiplexed traffic streams, that are leaky bucket regulated with peak rate, mean rate and burst size constraints. We consider independent, heterogeneous streams, which are multiplexed in a common buffer. We derive bounds on the tail of the probability distribution function of the buffer content in the stationary regime. We identify in particular the extremal traffic from the point of view of buffer overflow in the single input case. We then consider the case when the number of sources is large and the buffer is also scaled according to the number of streams. We finally show that an $M/G/1$ bound, where service times and the intensity of the input Poisson process depend on the (σ, ρ) parameters of each source, holds in the unscaled buffer case.

1 Introduction

The negotiation of traffic parameters prior to any data transfer has been a common approach followed by standardization bodies (ATM Forum, IETF, etc.) for guaranteeing quality of service (QoS) in broadband packet networks. Traffic sources are characterized by a restricted number of parameters, and the network operator uses these to provision sufficient resources in the network so as to guarantee the negotiated QoS level. Traffic parameters are enforced at network access point by means of simple mechanisms like leaky buckets in order to prevent any traffic violations and to protect well-behaved users.

The traffic parameters, which are the most commonly used, are the peak rate π, the mean rate ρ and the bucket size σ. A source conforming to the parameters (σ, ρ, π) is said to be (σ, ρ, π)-regulated. Such sources are the central objects of the so-called network calculus, originally developed by Cruz [1, 2] and more recently by Leboudec [3] and Chang [4]. The (σ, ρ, π) parameters are used to estimate the performance of the network, for instance in terms of delays. The main drawback of usual network calculus, however, is that the performance bounds correspond to worst-case bounds, which are very conservative for network resource allocation. Furthermore, they do not take into account the statistical independence of sources.

In this paper, we address the issue of estimating the overflow probability, when regulated sources are multiplexed in a buffer. For this purpose, we investigate estimating the tail of the buffer content distribution. Given the interest in providing QoS, this problem has received a lot of attention recently. In an early paper, Kesidis and Konstantopoulos [5] studied the problem of estimating the tail distribution in an infinite buffer fed with a (π, σ, ρ)-regulated source. They showed that the extremal traffic shape, which maximizes the fraction of time the

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buffer content is above a threshold b, is not the usual periodic On/Off process but is composed of burst at the peak rate followed by an activity period at rate c (the service rate of the buffer) and then a silent period. The duration of transmitting at the peak rate depends on threshold b and they used this characterization to derive an upper bound for the fraction of traffic lost in a buffered system with capacity b. They also addressed the problem of multiplexing many independent sources and derived a Chernoff bound in [6] based on bounding the buffer content by using separate queues.

Another approach to studying the superposition of regulated sources in a buffer has been presented by Busson and Massoulie [7], where they used Hoeffding's inequality based on the fact that the total number of packet arrivals in a given time interval from a regulated source is bounded. This approach was recently extended by Vojnovic and LeBoudec [8], where they also consider the so-called many sources asymptotics framework, which is a key regime when the peak rate of sources is small when compared with the server capacity and there are many sources. See [9], [10], and [11], for results on many sources asymptotics. Chang et. al [12] have also considered a different approach to address the many sources asymptotics using the fact that the moment generating function of any bounded random variable can be bounded by a Bernoulli r.v. taking values in $\{x_{min}, x_{max}\}$ with appropriate probabilities of being in the max (min) state to give the right mean.

In this paper, we continue along the same line of investigations to determine the loss performance, when independent regulated traffic streams are multiplexed in a buffer. Our motivation is due to the fact that several of the reported bounds have been derived under restrictive assumptions or else yield pessimistic bounds (as shown in the following). We consider a fluid infinite capacity queue fed with the superposition of (σ, ρ, π)-regulated traffic sources and for a given threshold b, we determine the worst case traffic, which maximizes the fraction of traffic arriving while the buffer content is greater than or equal to a given threshold b, referred to as freeze-out fraction, which serves as an estimate of the loss probability in the finite capacity case. This allows us to derive an upper-bound for the freeze-out fraction in the case of a single source system.

We show that while the bound on the freeze-out fraction is the same as that obtained in [5], the extremal source is slightly different. Indeed, it was remarked by Kesidis and Kostantopoulos in [5] that the extremal source they exhibited was not unique. However, our extremal characterization is important because it plays a crucial role in determining the many sources asymptotics for which tight bounds are derived. These bounds are different from those in [8] and easy to compute. They work well when the buffers are also scaled. Then, we consider another limit, which is related to increasing the number of sources while keeping the load fixed and assuming that the buffer is not scaled; this yields a different type of bound which is exact in the limit characterized by the tail of an $M/G/1$ queue, where service times and the intensity of the Poisson process depend on the parameters σ_i and ρ_i of the different sources.

The organization of this paper is as follows: The formulation of the problem and preliminary results are presented in Section 2. We then consider statistical multiplexing of many streams. We first obtain the results on the many sources asymptotic in Section 3. We subsequently propose in Section 4 another stochastic bound for unscaled buffers based on a different type of scaling. Numerical results are given in Section 5. Finally, concluding remarks are presented in Section 6.

2 Problem formulation and preliminary results

Consider N independent stationary fluid flow sources multiplexed in a single FIFO server queue drained at constant rate c and assume that flow $\#j$, $j = 1, \ldots, N$ is constrained by a $(\sigma_j, \rho_j, \pi_j)$ traffic descriptor, where σ_j, π_j and ρ_j are the bucket size, the peak rate, and the mean rate, respectively. The $(\sigma_j, \rho_j, \pi_j)$ constraint for source $\#j$ consists of assuming that the amount of data, which can be generated by source $\#j$ over the time interval $[s, t)$, denoted by $A_j(s, t)$, is such that

$$A_j(s,t) \leq \min\{\pi_j(t-s), \sigma_j + \rho_j(t-s)\}. \tag{1}$$

In the following, we denote by w_t the workload (or equivalently the amount of fluid) in the buffer at time t and by $\alpha(t)$ the arrival curve, i.e., the total amount of fluid, which can arrive at the buffer in the time interval $[0, t]$; $\alpha(t)$ is such that

$$\alpha(t) \leq A(0,t) \stackrel{def}{=} \sum_{j=1}^{N} A_j(0,t).$$

Throughout this paper, we assume that the queue load $\rho/c < 1$, where $\rho = \sum_{j=1}^{N} \rho_j$, so that the system is stable and a stationary regime exists.

The objective of this paper is to derive an upper bound for the bit loss probability, when the capacity of the buffer into which the N flows are multiplexed is finite. As a first approximation, this quantity is upper bounded by the fraction of bits which enter the infinite capacity system, while the workload exceeds the threshold b. In the following, we are specifically interested in establishing an upper bound for this latter quantity, referred to as freeze-out fraction and formally defined by

$$\mathcal{P} \stackrel{def}{=} \lim_{t\to\infty} \frac{1}{A(0,t)} \int_0^t \mathbb{1}_{\{w_s \geq b\}} A(ds).$$

Since $A(ds) = \dot{w}_s + c$, when the queue is not empty (with the notation $\dot{w}_t = dw/dt$), we deduce that the above quantity is simply equal to $c\mathbb{P}\{w_0 \geq b\}/\rho$, where $\mathbb{P}\{w_0 \geq b\}$ is the probability that the workload in the queue exceeds b in the stationary regime. In other words, $\mathcal{P} = \mathbb{P}\{w_0 \geq b \mid w_0 > 0\}$.

Several upper bounds for $\mathbb{P}\{w_0 \geq b\}$ have been derived in the literature when sources are homogeneous, i.e, characterized by the same triplet (σ, ρ, π) [13]. Considering N (σ, ρ, π)-regulated sources, Kesidis and Kostantopoulos [5] have shown that

$$\mathbb{P}\{w_0 > b\} \leq \exp\left(-N\frac{b}{b_{req}}\log\left(\frac{b}{\rho D_{max}}\right) + N\left(1 - \frac{b}{b_{req}}\right)\log\left(\frac{b_{req} - \rho D_{max}}{b_{req} - b}\right)\right), \tag{2}$$

for $N\rho D_{max} \leq b \leq b_{req}$, where b_{req} is the buffer capacity required for loss free operation, given in the present case by

$$b_{req} = \frac{(N\pi - c)\sigma}{\pi - \rho},$$

and D_{max} is the worst case delay in the loss free system, given by

$$D_{max} = \frac{(N\pi - c)\sigma}{(\pi - \rho)c}.$$

In the same situation, letting τ be the maximum busy period, given by $\tau = N\sigma/(c - N\rho)$, Chang *et al* [12] have derived the other upper bound :

$$\mathbb{P}\{w_0 > b\} \leq \sum_{k=0}^{K-1} \exp\left(-N g_k\right), \tag{3}$$

where g_k is given by

$$g_k = \begin{cases} +\infty \quad \text{if} \quad b > \alpha(s_k) - cs_k, \\ 0 \quad \text{if} \quad b < N\rho s_{k+1} - cs_k, \\ \frac{cs_k+b}{\alpha(s_{k+1})} \log\left(\frac{cs_k+b}{N\rho s_{k+1}}\right) + \left(1 - \frac{cs_k+b}{\alpha(s_{k+1})}\right) \log\left(\frac{\alpha(s_{k+1})-cs_k-b}{\alpha(s_{k+1})-N\rho s_{k+1}}\right) \quad \text{otherwise.} \end{cases}$$

α is the arrival curve and $\underline{s} = (s_0 = 0, s_1, \ldots, s_{K_1}, s_K = \tau)$ is any partition of the interval $[0, \tau]$. As in [12], one may take $s_k = k$, but as mentioned in [13], a better bound is obtained by taking the minimum over all possible partitions $\underline{s}$.

The objective of this paper is to obtain bounds, which are not only more accurate than those cited above but also easier to manipulate for network dimensioning purposes. But, before proceeding with this task, let us note that in the case of a single source, an upper bound for the freeze-out fraction $\mathcal{P}$ has been obtained by Kesidis and Kostantopoulos [6].

Theorem 1 (Kesidis and Kostantopoulos [6]) *Under the assumptions $\pi > \rho > c$ and $(\pi - c)\sigma/(\pi - \rho) > b$, the freeze-out fraction $\mathcal{P}$ in the single server queue fed with a (σ, ρ, π)-regulated fluid traffic source is upper bounded as follows:*

$$\mathcal{P} \leq \frac{\sigma - \frac{\pi-\rho}{\pi-c} b}{\sigma - \frac{\rho}{c} b} \stackrel{def}{=} \mathcal{P}_{\max}. \tag{4}$$

Kesidis and Kostantopoulos [*ibid.*] also exhibited a worst case traffic pattern, that achieves the upper-bound (4). As noted by those authors, the worst case traffic pattern is not unique and one may also note that their worst-case traffic has been designed so as to maximize the freeze-out fraction for a given threshold b. In the following, we exhibit another worst case traffic pattern, which has the advantage of maximizing the quantity of information generated over each activity period. This worst case traffic also turns out to be natural in the subsequent analysis involving many sources.

First note that the freeze-out fraction $\mathcal{P}(\tau)$ over a busy period with length τ and starting at time 0 is defined as

$$\mathcal{P}(\tau) = \frac{1}{c\tau} \int_0^\tau \mathbb{1}_{\{w_s \geq b\}} A(ds). \tag{5}$$

We have the following result.

Lemma 1 *In the case of a single source, the traffic pattern, which maximizes the freeze-out fraction $\mathcal{P}(\tau)$ over a busy period with length τ such that*

$$\tau \geq \tau_{\min}(b) \stackrel{def}{=} \frac{\pi b}{c(\pi - c)}, \tag{6}$$

is defined as follows:

- *If $\tau \leq (\pi\sigma)(c(\pi-\rho))$, the extremal traffic pattern is periodic and composed of a burst at the peak rate with duration $c\tau/\pi$, followed by a silence period with duration $(c-\pi)\tau/\pi$.*

- *If $\pi\sigma/(c(\pi-\rho)) \leq \tau \leq \tau_{\max}$, the extremal traffic is periodic and composed of a burst at the peak rate π with length $\sigma/(\pi-\rho)$, followed by an activity period at rate ρ with length*

$$\frac{c}{\rho}\left(\tau - \frac{\pi\sigma}{c(\pi-\rho)}\right),$$

and followed in turn by a silence period with length $(c-\rho)\left(\frac{\sigma}{c-\rho}-\tau\right)/\rho$.

Proof. In a first step, we note that the process $\{w_t\}$ can hit the level b over the busy period with length τ if and only if τ satisfies inequality (6). Indeed, the smallest time at which the process can reach level b is obtained when the input rate is equal to the peak rate π, and this smallest time is equal to $t'_b = b/(\pi-c)$. Then, the smallest busy period is obtained when the source remains silent after time t'_b. Thus, the length of smallest busy period over which level b can be reached is equal to $\tau_{\min}(b)$ defined by equation (6).

Assuming that inequality (6) holds, we consider a busy period with length τ and starting at time 0. What we have to determine is the traffic pattern, which maximizes the quantity $\mathcal{P}(\tau)$ defined by equation (5). In other words, we have to find a realization $w = \{w_t\}_{t\in[0,\tau]}$ so that $\mathcal{P}(\tau)$ is maximal.

The function w has to be in the set $\mathcal{Y}$ of admissible solutions, which is given, owing to the (σ,ρ,π)-constraint, by

$$\mathcal{Y} = \{w \in \mathcal{C}^1_p[0,\tau] : w_0 = w_\tau = 0, \text{ and } 0 \leq w_t \leq w^*_t,\ 0 \leq t \leq \tau\},$$

where $\mathcal{C}^1_p[0,\tau]$ is the set of functions, which are continuous over $[0,\tau]$ and piecewise derivable over $(0,\tau)$, and the function $t \to w^*_t$ is defined as follows:

- if $\tau \leq \pi\sigma/(c(\pi-\rho))$,

$$w^*_t = \begin{cases} (\pi-c)t,\ 0 \leq t \leq t'_1 \stackrel{def}{=} c\tau/\pi, \\ c(\tau-t),\ t'_1 \leq t \leq \tau. \end{cases} \tag{7}$$

- if $\tau \geq \pi\sigma/(c(\pi-\rho))$,

$$w^*_t = \begin{cases} (\pi-c)t,\ 0 \leq t \leq t_1 \stackrel{def}{=} \frac{\sigma}{\pi-\rho}, \\ \sigma + (\rho-c)t,\ t_1 \leq t \leq t_2 \stackrel{def}{=} \frac{(c\tau-\sigma)}{\rho}, \\ \sigma + \rho t_2 - ct,\ t_2 \leq t \leq \tau, \end{cases} \tag{8}$$

For $w \in \mathcal{Y}$, it is easily checked that

$$\mathcal{P}(\tau) = J(w) \stackrel{def}{=} \frac{1}{\tau}\int_0^\tau \mathbb{I}_{\{w_t \geq b\}}dt,$$

and we see that the optimization problem under consideration consists of finding an element w of $\mathcal{Y}$, which maximizes the time spent above b during the busy period.

Let us define on $\mathcal{Y}$ the partial order $\preceq$ as follows:

$$w \preceq v \quad \text{iff} \quad w_t \leq v_t \text{ for all } 0 \leq t \leq \tau.$$

It is easily seen that $J(w) \leq J(v)$ if $w \preceq v$ and then that the functional J is monotonic increasing with respect to the partial order $\preceq$.

The element w^* is extremal in $\mathcal{Y}$ in the sense that every $w \in \mathcal{Y}$ is such that $w \preceq w^*$. Indeed, in the case $\tau \leq \pi\sigma/(c(\pi-\rho))$ (resp. $\tau \geq \pi\sigma/(c(\pi-\rho))$), owing to the (σ, ρ, π) constraint, $w_t \leq w^*_t$ for all $t \in [0, t'_1]$ (resp. $t \in [0, t_2]$). Now, assume that there exists some $t_0 \in [t'_1, \tau]$ (resp. $t_0 \in [t_2, \tau]$) such that $w_{t_0} > w^*_{t_0}$. Then, from the Mean Value Theorem, there exists some $t'_0 \in [t_0, \tau]$ such that $w_{t_0} = -(\tau - t_0)\dot{w}_{t'_0} > w^*_{t_0} = c(\tau - t_0)$, which implies that $\dot{w}_{t'_0} < -c$. This latter inequality is not possible since the drain rate from the queue cannot exceed c. As a consequence, for every $w \in \mathcal{Y}$, we have $w \preceq w^*$. Since the functional J is increasing, the element w^* is extremal.

Now, returning to the input process, when $\tau \leq \pi\sigma/(c(\pi-\rho))$, the input process, which maximizes the freeze-out fraction in the busy period with length τ is the classical On/Off process; during the On period the arrival rate is equal to the peak rate and the length of the On period is equal to $t'_1 = c\tau/\pi$. This is the classical result stating that the optimal control is "bang-bang".

In the case when $\tau \geq \pi\sigma/(c(\pi-\rho))$, the input process, which realizes the optimal trajectory w^* over a busy period, is composed of a burst at the peak rate π and with duration t_1, followed by an activity period at rate ρ with length $t_2 - t_1$, and then by a silence period with length S given by

$$S = \tau - t_2 = \frac{c-\rho}{\rho}\left(\frac{\sigma}{c-\rho} - \tau\right). \tag{9}$$

Note that S is positive if and only if $\tau < \sigma/(c-\rho)$. The length of the busy period of a queue with an input process satisfying a (σ, ρ, π)-constraint is thus necessarily upper -bounded by $\sigma/(c-\rho)$. This completes the proof. $\square$

By using Lemma 1, we deduce that for a busy period with length τ, $\mathcal{P}(\tau)$ is upper bounded by $\mathcal{P}^*(\tau)$ which corresponds to the freeze-out fraction obtained for the extremal function w^*. Simple computations show that this quantity is given by:

$$\mathcal{P}^*(\tau) = \begin{cases} 0 \quad \tau \leq \pi b/(c(\pi - c)) \\ 1 - \dfrac{\pi b}{c\tau(\pi - c)} \quad \pi b/(c(\pi-c)) \leq \tau \leq \tau^*_b \\ \\ \mathcal{P}(\tau^*_b)\dfrac{\tau^*_b}{\tau} \quad \tau^*_b \leq \tau \leq \tau_{\max}, \end{cases} \tag{10}$$

where the critical length of the busy period τ^*_b is given by $\tau^*_b = (c\sigma - \rho b)/(c(c-\rho))$, and the quantity $\mathcal{P}(\tau^*_b)$ is given by $\mathcal{P}(\tau^*_b) = (\sigma - \frac{\pi-\rho}{\pi-c}b)/(\sigma - \frac{\rho}{c}b)$. It turns out from the above equation that for any busy period with length τ, $\mathcal{P}(\tau) \leq \mathcal{P}^*(\tau^*_b)$, which entails equation (4). The worst case traffic pattern, which achieves the upper bound $\mathcal{P}(\tau^*_b)$ is as specified in Lemma 1 for $\tau = \tau^*_b$. This is another worst case traffic pattern, which maximizes the fraction of time the buffer content is above b. Moreover, we have

$$\mathbb{P}\{w \geq b \mid w > 0\} \leq \mathcal{P}_{\max}.$$

3 Many sources asymptotics

We now consider the problem of estimating the freeze-out fraction, when there is a large number (say, proportional to some N with large N) of independent traffic streams. When the transmission capacity c is large, and in particular, $c/\pi = O(N)$, this puts us in the regime of the many sources asymptotics, which have been studied by many authors, see for instance [9, 10, 11]. We use the results and formalism developed in Likhanov and Mazumdar [11], which can be readily extended to the continuous-time case by a discretization argument, see for example the papers by Mandjes and Kim [14] and Guibert and Simonian [15], where they assume a locally convex behavior of a rate function (see equation (12) below).

We consider the heterogeneous case of M types of sources assuming that there are $O(N)$ numbers of sources of each type in the system. In particular, let there be $Nn_i,\ i = 1, \ldots, M$ sources of each type, which are statistically independent. The sources access a buffer drained by a server of large capacity Nc. It is assumed that $\sum_{i=1}^{M} n_i\pi_i > c$, otherwise overflow cannot occur. Let $A^N(0,t) = \sum_{i=1}^{M}\sum_{j=1}^{Nn_i} A_{j,i}(0,t)$ denote the total arrival curve for the buffer.

It is assumed that $\mathbb{E}[A^N(0,1)] = \sum_{i=1}^{M}\sum_{i=1}^{N} n_i\rho_i < Nc$ implying the system is stable and that there exists a stationary version of the workload (buffer content or unfinished work). Let W_0^N denote the workload in the queue at an arbitrary instant in the stationary regime. From Reich's formula, we have

$$W_0^N = \sup_{t\geq 0}\{A^N(-t,0) - Nct\}. \tag{11}$$

Let $\phi_{i,t}(h) = \mathbb{E}[e^{hA_i(0,t)}]$ denote the moment generating function associated with source i where $h > 0$ and define:

$$I_t(a) = \sup_{h\geq 0}\{ah - \sum_{i=1}^{M} n_i \ln(\phi_{i,t}(h))\}. \tag{12}$$

$I_t(a)$ is called the rate function associated with $A(.)$.

We now state the main result regarding the stationary tail distribution, which has been shown in [11] for the discrete-time case but which can be extended to the continuous-time case via a discretization argument and assuming continuity of $I_t(a)$ w.r.t. a. To simplify the notation, we sometimes use $A^N(0,t)$ by A_t^N in the following.

Proposition 1 *Assume that there exists a unique $t_0 < \infty$ such that:*

$$I_{t_0}(ct_0 + b) = \min_{t\geq 0} I_t(ct + b) > 0 \tag{13}$$

Suppose that $\liminf_{t\to\infty} I_t(ct+b)/\log t > 0$ and that $I_t(x)$ is continuous in x. Then, as $N \to \infty$,

$$\mathbb{P}(W_0^N > Nb) = \frac{e^{-NI_{t_0}(ct_0+b)}}{\tau\sqrt{2\pi\kappa^2 N}}\left(1 + O\left(\frac{1}{N}\right)\right), \tag{14}$$

where $I_t(.)$ is defined in equation (12) and τ is the unique solution to the equation

$$\frac{\sum_{i=1}^{M} n_i\phi'_{i,t_0}(\tau)}{\phi_{i,t_0}(\tau)} = ct_0 + b, \tag{15}$$

and

$$\kappa^2 = \left(\sum_{i=1}^{M} \frac{n_i\phi''_{i,t}(\tau)}{\phi_{i,t}(\tau)} - \left(\sum_{i=1}^{M} n_i\frac{\phi'_{i,t}(\tau)}{\phi_{i,t}(\tau)}\right)^2\right). \tag{16}$$

The important point to note is the existence of t_0, which denotes the critical or most likely time-scale to overflow. We shall now exploit this key result to derive the worst-case traffic source conforming to the (ρ, σ, π) bound in the case of many sources. This is given in the proposition below.

Proposition 2 *Let $\int_0^t r_{i,s} ds = A_i(0,t) \leq \min\{\pi_i t, \rho_i t + \sigma_i\}$ with $\sum_{i=1}^M n_i \rho_i < c$. Let*

$$A^N(0,t) = \sum_{i=1}^{M} \sum_{j=1} N n_i A_{i,j}(0,t)$$

denote the aggregate flow of $N \sum_{i=1}^M n_i$ mutually independent sources. Then, the extremal source is periodic and given by:

$$r^*_{i,t} = \begin{cases} \pi_i \ , & 0 \leq t \leq \sigma_i/(\pi_i - \rho_i), \\ \rho_i, & \sigma_i/(\pi_i - \rho_i) < t < t_0, \\ 0, & t_0 < t \leq t_0 + \sigma_i/\rho_i, \end{cases}$$

where t_0 is the most-likely time-scale to overflow for the $N \sum_{i=1}^M n_i$ independent sources of the type above.

Proof. Let W_0^N denote the stationary buffer-content. From stationarity $A^N(-t,0) \stackrel{d}{=} A^N(0,t)$. From the proof of the many-sources asymptotics in [11], it follows that:

$$\mathbb{P}(W_0^N > Nb) = \mathbb{P}\left(A_{t_0}^N > N(Ct_0 + b)\right) \times \left(1 + O(e^{-\varepsilon N})\right).$$

for some $\varepsilon > 0$.

In the following we take t_0 to be fixed. We now obtain the contribution of the type i sources to the bound. Define

$$A_t^{N_i -} = A_t^N - \sum_{j=1}^{Nn_i} A_j(0,t),$$

i.e., we consider all inputs except the type i inputs. Then, denoting by $c_t = ct + b$, we have

$$\mathbb{P}(A_{t_0}^N > Nc_{t_0}) = \int_0^\infty \mathbb{P}(A_{t_0}^{N_i -} > Nc_{t_0} - y) \times d\mathbb{P}(A_{i,t_0}^N \leq y). \tag{17}$$

From Bahadur-Rao theorem [16], we obtain:

$$\mathbb{P}\left(A_{t_0}^{N_i -} > Nc_t - y\right) = K(N) e^{\tau y} \left(1 + O(\frac{1}{N}\right),$$

where $K(N)$ is a term (indeed it is just given by equation (14)), which does not depend on y, and τ satisfies:

$$\sum_{j=1, j \neq i}^{M} \frac{\phi_{j,t_0}(\tau)}{\phi'_{j,t_0}(\tau)} = Ct_0 + b.$$

Hence, up to an error factor of $\left(1 + O(\frac{1}{N})\right)$, we obtain

$$\mathbb{P}\left(W_0^N > Nb\right) = K(N) \int_0^\infty e^{\tau y} d\mathbb{P}(A_i^N(0,t_0) \leq y))$$

and then,

$$\mathbb{P}\left(W_0^N > Nb\right) = K(N)\left(\mathbb{E}[e^{\tau A_i(0,t_0)}]\right)^{Nn_i} = K(N)\left(\phi_{t_0}(\tau)\right)^{Nn_i}.$$

Now from the ergodicity of the source:

$$\phi_{t_0}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{\tau A_i(s,s+t_0)}ds.$$

Hence it is clear that in order to bound the overflow probability, we have to maximize the r.h.s. of equation (17). Hence, we need determine the extremal traffic pattern, which maximizes $\mathbb{E}[e^{\tau A_i(0,t_0)}]$. Following exactly the same arguments as in the proof of Lemma 1, it is easy to prove the result, noting that by the above source is extremal over the interval $[0, t_0]$ in the sense that it maximizes the quantity of information over a given period. This completes the proof. $\square$

Using the above result we can now state the main result for the buffer overflow bound.

Proposition 3 *Consider a fluid queueing system with server capacity Nc and an infinite buffer which is accessed by Nn_i, $i = 1, 2 \ldots, M$ independent sources, where the total volume of fluid transmitted by a source of type i in $(0, t]$ satisfies:*

$$A_i(0,t) \le \min\{\pi_i t, \rho_i t + \sigma_i\}.$$

Assume that $\mathbb{E}[A_i(0,1)] = \rho_i$ and $\sum_{i=1}^M n_i\rho_i < c$. Then, the tail distribution of the stationary buffer content satisfies:

$$\mathbb{P}(W_0^N > Nb) = \frac{e^{-NI_{t_0}(ct_0+b)}}{\tau\sqrt{2\pi\kappa^2 N}}\left(1 + O\left(\frac{1}{N}\right)\right), \tag{18}$$

where the quantities $I_{t_0}(ct_0 + b)$, τ and σ are calculated as follows:

- *Define*

$$\phi_{i,t}(\tau) = \frac{1}{t+\frac{\sigma_i}{\rho_i}}\int_0^{t+\frac{\sigma_i}{\rho_i}} e^{\tau\int_u^{u+t} r^*_{i,s}ds}du. \tag{19}$$

- *Compute:*

$$I_{t_0}(ct_0+b) = \inf_t \sup_h \left\{(ct+b)h - \sum_{i=1}^M n_i \ln(\phi_{i,t}(h))\right\}. \tag{20}$$

- *Compute τ as the solution to:*

$$\sum_{i=1}^M n_i \frac{\phi'_{i,t_0}(\tau)}{\phi_{i,t_0}(\tau)} = ct_0 + b. \tag{21}$$

- *Finally compute*

$$\kappa^2 = \sum_{i=1}^M n_i\frac{\phi''_{i,t_0}(\tau)}{\phi_{i,t_0}(\tau)} - \left(\sum_{i=1}^M n_i\frac{\phi'_{i,t_0}(\tau)}{\phi_{i,t_0}(\tau)}\right)^2. \tag{22}$$

Note that the moment generating function for any source of type i will not involve time intervals beyond the amount $t_0 + \sigma_i/\rho_i$ since the average loss is obtained by a periodic input with randomized phase.

While the calculations seem cumbersome, the procedure is more explicit than the one proposed in [8]. The principal difference is that in our approach, we explicitly take into account the many sources effect and determine the rate function for the source, which is extremal for the overflow asymptotics rather than a priori first bounding the probability and then trying to make the bound small as it is done in the Hoeffding type of argument [8, 12].

The validity of the many sources asymptotic is when a large number of sources are multiplexed and the buffer grows correspondingly. However, in many applications, when sources can use significant portions of the bandwidth and the buffers do not scale, we need another type of bound. This is discussed in the next section.

4 Stochastic bounds

Instead of establishing an asymptotic result, when the number of sources is large, we derive in this section a stochastic bound, which relies on stochastic ordering arguments. For this purpose, we use a stochastic domination property enjoyed by a FIFO queue fed with the superposition of leaky bucket regulated traffic sources.

Let w denote the content in the stationary regime of a buffer fed with a (σ, ρ, π)-regulated source; the buffer is drained at constant rate c. Let $\tilde{w}$ denote the content in the stationary regime of a buffer drained at constant rate c and fed with batches with size σ arriving according to a Poisson process with intensity ρ/σ. The Laplace transform $\mathbb{E}[e^{\xi\tilde{w}}]$ is given by

$$\mathbb{E}[e^{-\xi\tilde{w}}] = \frac{(c-\rho)\xi}{c\xi + \frac{\rho}{\sigma}e^{-\sigma\xi} - \frac{\rho}{\sigma}}.$$

Let $\tilde{\xi}_0$ denote the module of the pole with the smallest module of the above Laplace transform. $\tilde{\xi}_0$ can be written as $\tilde{\xi}_0 = \frac{1}{\sigma}\eta(\frac{c}{\rho})$, where for $x > 1$, $\eta(x)$ is the root not equal to 0 and with the smallest module of the equation

$$-x\eta + e^{\eta} - 1 = 0. \tag{23}$$

η is real, positive, and an increasing function of x. Moreover, note that

$$\eta > 1 - 1/x. \tag{24}$$

Indeed, if $\eta \leq 1 - 1/x$, by using the fact that $e^{\eta} < 1 + \eta/(1-\eta)$ for $\eta < 1$, we would have $\eta > 1 - 1/x$, which is in contradiction with the fact that $\eta < 1 - 1/x$. In addition, straightforward manipulations show that

$$\eta e^{\eta} > 2(e^{\eta} - x) \tag{25}$$

From the queueing point of view, we have the important domination property (referred to as Kingman's bound):

$$\mathbb{P}\{\tilde{w} \geq b\} \leq e^{-\tilde{\xi}_0 b}.$$

$\tilde{\xi}_0$ is referred to as Kingman's exponent in the following. In a first step, we prove the following technical lemma.

Lemma 2 *We have $w \leq_{st} \tilde{w}$, i.e., for all $b \geq 0$, $\mathbb{P}\{w \geq b\} \leq \mathbb{P}\{\tilde{w} \geq b\}$.*

Proof. By using Theorem 1, we know that for $b \leq \sigma$ and $c = 1$

$$\mathbb{P}\{w \geq b \mid w > 0\} \leq \frac{\sigma - \frac{\pi-\rho}{\pi-c} b}{\sigma - \frac{\rho}{c} b} \leq \frac{\sigma - b}{\sigma - \frac{\rho}{c} b},$$

where the last inequality is obtained by letting $\pi \to \infty$. Now, by using a classical result by Erlang (see [17]), we know that over the interval $[j\sigma, (j+1)\sigma]$ for $j \geq 0$, we have

$$\mathbb{P}(\tilde{w} \leq x) = \left(1 - \frac{\rho}{c}\right) \sum_{i=0}^{j} \frac{(i - x/\sigma)^i}{i!} \left(\frac{\rho}{c}\right)^i e^{-\frac{\rho}{c}(i - x/\sigma)}.$$

For $b \in [0, \sigma]$, it is easily checked that

$$\frac{\sigma - b}{\sigma - \frac{\rho}{c} b} \leq \frac{c}{\rho} \left(1 - \left(1 - \frac{\rho}{c}\right) \exp\left(\frac{\rho b}{c\sigma}\right)\right).$$

since $\frac{\rho b}{c\sigma} < 1$ and then, $\exp(\frac{\rho b}{c\sigma}) \leq \sigma/(\sigma - b\frac{\rho}{c})$. Hence, for all $b \geq 0$, $\mathbb{P}\{w \geq b \mid w > 0\} \leq \mathbb{P}\{\tilde{w} \geq b \mid \tilde{w} > 0\}$. This completes the proof. □

Let us consider now a buffer drained at rate c and fed with the superposition of N independent regulated traffic sources; source $\#j$ is $(\sigma_j, \rho_j, \pi_j)$-regulated. Let W denote the buffer content in the stationary regime. Then, we have the following result.

Theorem 2 *The buffer content W in the stationary regime is such that*

$$\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}\{W \geq x\} \leq -\tilde{\xi}, \tag{26}$$

where $\tilde{\xi}$ is the Kingman's exponent of the $M/G/1$ queue, where the intensity of the input Poisson process is $\sum_{i=1}^{N} \rho_i/\sigma_i$ and the distribution of the service time S of a customer is given by

$$\mathbb{P}\{S = \sigma_i\} = \frac{\rho_i/\sigma_i}{\sum_{j=1}^{N} (\rho_j/\sigma_j)}, \quad i = 1, \ldots, N. \tag{27}$$

Proof. Let us consider a partition $\{c_i\}$, $i = 1, \ldots, N$, of the server capacity c such that $\sum_{i=1}^{N} c_i = c$ and $c_i > \rho_i$ for $i = 1, \ldots, N$. From Reich's formula, we have

$$W = \sup_{t \geq 0} \sum_{i=1}^{N} (A_i(0,t) - c_i t) \leq \sum_{i=1}^{N} w_i,$$

where w_j is the content in the stationary regime of a buffer drained at constant rate c_j and fed with source j, which is $(\sigma_j, \rho_j, \pi_j)$-regulated. Hence, for all $b \geq 0$, we have by using Lemma 2

$$\mathbb{P}\{W \geq x\} \leq \mathbb{P}\left\{\sum_{i=1}^{N} w_i \geq x\right\} \leq \mathbb{P}\left\{\sum_{i=1}^{N} \tilde{w}_i \geq x\right\},$$

where $\tilde{w}_j$ is the content in the stationary regime of a fluid buffer drained at constant rate c_j and fed with batches of fluid with size σ_j arriving according to a Poisson process with rate

ρ_j/σ_j. Let $\tilde{\xi}_j(c_j)$ denote the Kingman's exponent of this queue; $\tilde{\xi}_j(c_j) = \frac{1}{\sigma_j}\eta(\frac{c_j}{\rho_j})$, where η is defined by equation (23).

Since the inputs are by assumption independent, the random variables $\tilde{w}_i$ are independent and the Laplace transform

$$\mathbb{E}\left[e^{-\xi\sum_{i=1}^N \tilde{w}_i}\right] = \prod_{i=1}^N \mathbb{E}\left[e^{-\xi\tilde{w}_i}\right]. \tag{28}$$

The tail of the probability distribution function of the random variable $\sum_{j=1}^N \tilde{w}_j$ is governed by the root with the smallest module of the above Laplace transform; this root is $-\inf_i\{\tilde{\xi}_i(c_i)\}$. This property holds for any partition $\{c_i\}$ of the server capacity such that $c_i > \rho_i$ for all $i = 1, \ldots, N$.

To determine the exact tail behavior of the probability distribution of the random variable W, we are led to determine the maximum value of $\inf\{\tilde{\xi}_i(c_i)\}$ over all the partitions of the server capacity c such that for all $i = 1, \ldots, N$, $c_i > \rho_i$. Since $\xi(c_i) = \frac{1}{\sigma_i}\eta(\frac{c_i}{\rho_i})$ and $\eta(x)$ is an increasing function of x, the optimal values c_i^* are such that all the values $\frac{1}{\sigma_i}\eta(\frac{c_i^*}{\rho_i})$ for $i = 1, \ldots, N$ are equal to some constant, say ξ^*. Indeed, if all the $\xi_j(c_j)$ were not equal, it would always be possible to increase the maximum value of the minimum by decreasing the largest value. ξ^* is then the smallest solution to the equations:

$$i = 1, \ldots N, \; \frac{c_i^*}{\rho_i}\sigma_i\xi^* = e^{\sigma_i\xi^*} - 1 \quad \Rightarrow \quad c\xi^* = \sum_{i=1}^N \frac{\rho_i}{\sigma_i}\left(e^{\xi^*\sigma_i} - 1\right)$$

It turns out that ξ^* is the Kingman's exponent $\tilde{\xi}$ of the $M/G/1$ queue where the intensity of the input Poisson process is $\sum_{i=1}^N \rho_i/\sigma_i$ and the distribution of the service time of a customer is given by equation (27).

Since all the parameters $\tilde{\xi}_j(c_j^*)$ are equal, the point $-\tilde{\xi}$ is a pole with order N for the Laplace transform (28). The Laplace transform $\mathbb{E}[e^{-\xi\tilde{w}_i}]$ can specifically be written as

$$\mathbb{E}\left[e^{-\xi\tilde{w}_i}\right] = a_i\left[\frac{\tilde{\xi}}{\xi+\tilde{\xi}} - \frac{c_i^* - \rho_i e^{\sigma_i\tilde{\xi}} + \tilde{\xi}\frac{\rho_i}{\sigma_i}\sum_{n=2}^\infty \frac{(-\sigma_i)^n}{n!}(\xi+\tilde{\xi})^{n-2}e^{\sigma_i\tilde{\xi}}}{c_i^* - \rho_i e^{\sigma_i\tilde{\xi}} + \frac{\rho_i}{\sigma_i}\sum_{n=2}^\infty \frac{(-\sigma_i)^n}{n!}(\xi+\tilde{\xi})^{n-1}e^{\sigma_i\tilde{\xi}}}\right],$$

where

$$a_i = \frac{c_i^* - \rho_i}{\rho_i e^{\sigma_i\tilde{\xi}} - c_i^*},$$

and it then follows that the Laplace transform (28) can be written as

$$\mathbb{E}[e^{-\xi\sum_{i=1}^N \tilde{w}_i}] = \left(\prod_{j=1}^N a_j\right)\left[\sum_{j=1}^N \kappa_j\left(\frac{\tilde{\xi}}{\xi+\tilde{\xi}}\right)^j + g(\xi)\right]$$

where the function g has poles with modules greater than $-\tilde{\xi}$ and κ_j is the coefficient of Y^j in the product $\prod_{j=1}^N(Y + b_j)$ with

$$b_j = \frac{\tilde{\xi}\rho_j\sigma_j e^{\sigma_j\tilde{\xi}}}{2(\rho_j e^{\sigma_j\tilde{\xi}} - c_j^*)} - 1.$$

Note that by using inequalities (24) and (25), it is easily checked that $b_j > 0$ and $a_j > 0$.

By using [18, Theorem 10.7], it follows that

$$\mathbb{P}\left\{\sum_{j=1}^{N} \tilde{w}_j \geq x\right\} \sim \left(\prod_{j=1}^{N} a_j\right)\left(\sum_{j=1}^{N} \frac{\kappa_j}{(j-1)!}\Gamma(j, x\tilde{\xi})\right),$$

where we have used the incomplete Gamma function $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$. By using the asymptotic equivalent $\Gamma(a,z) \sim z^{a-1}e^{-z}$ for large $z > 0$, we obtain

$$\mathbb{P}\left\{\sum_{j=1}^{N} \tilde{w}_j \geq x\right\} \sim \frac{\kappa_N}{(N-1)!}\left(\prod_{j=1}^{N} a_j\right)(x\tilde{\xi})^{N-1}e^{-x\tilde{\xi}}. \tag{29}$$

when $x \to \infty$. Taking logarithms and dividing by x, equation (26) follows. This completes the proof. $\square$

To conclude this section, note that the factor in front of the exponential term in equation (29) tends to 0 when $N \to \infty$, and then the tail of the probability distribution function of $\sum_{j=1}^{N} \tilde{w}_j$ is dominated by $\exp(-\tilde{\xi}x)$. The $M/G/1$ queue asymptotically yields a stochastic upper bound for the content of a buffer fed with the superposition of regulated inputs.

5 Numerical results

In a first step, we compare the bound given by equation (18) with the bounds proposed by [6] as given in [8]. Results are displayed in Figure 1, where we have only computed the rate functions associated with each bound and where we have indicated both the Chernoff bound as well as the more exact bound with the Gaussian multiplicative factor. Numerical evidence indicates that our bound is much tighter than that proposed by Kesidis and Kostantopoulos.

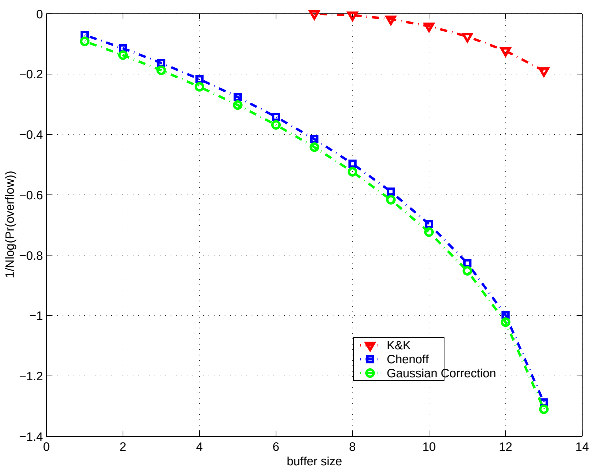
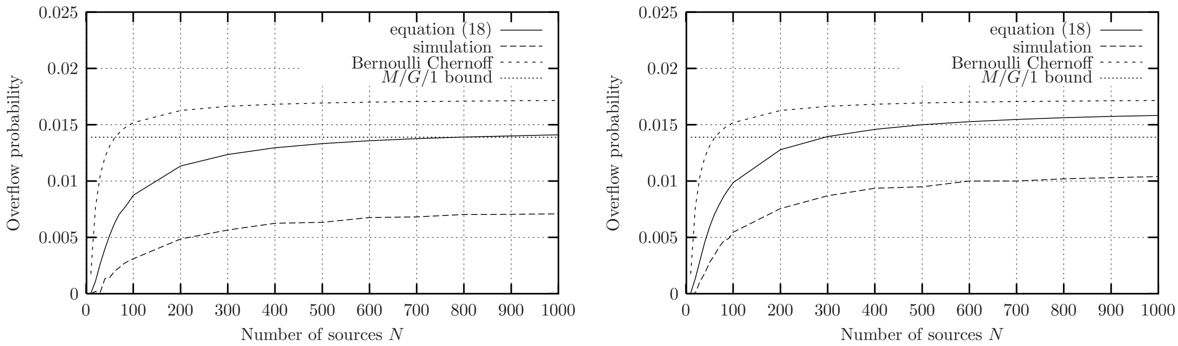


Figure 1: Rate functions associated with the Kesidis-Konstantopoulos (K&K) bound and many sources asymptotics bound (18), when sources are identical(i.e. $M = 1$) with parameters $\sigma = 0.05$, $\rho = 0.7$, $c = N \times 1$, $N = 50$, and $\pi = 5$

Figure 2 shows the many sources bound derived in equation (18) and the Bernoulli bound obtained via using the Hoeffding inequality for bounding the moment generating function as given by [12]. All these bounds are compared against simulations. It clearly appears that the many-sources bound becomes quite accurate as the number of sources grows. Also, we have indicated the bound obtained, when using the $M/G/1$ described in Theroem 2. Instead of using the exact estimate (26), we have used the value $\mathbb{P}\{\tilde{W} > b\} \sim (c-\rho)\tilde{\xi}\exp(-\tilde{\xi}b)/(\rho\exp(\sigma\tilde{\xi})-c)$, where $\tilde{W}$ denotes the workload in that queue. In the case considered, $\tilde{\xi} = 13.5094$.



(a) $\pi = 0.5$

(b) $\pi = 2$

Figure 2: Comparison of many sources bound, Bernoulli bound, and $M/G/1$ bound vs. simulation for identical sources, $c = 1$, $b = 0.3$, $\sigma = 0.05$, $N\rho = 0.7$.

6 Conclusion

In this paper, we obtained bounds on the tail distributions of the buffer content, when leaky bucket regulated traffic streams are multiplexed in a FIFO manner. The approach was to identify the worst case traffic streams, which conform with the (π, ρ, σ) envelope. Numerical results indicate that the bounds function quite well and indicate that substantial multiplexing gains are achievable.

Our many sources asymptotic bound is much tighter than the approximations given in the literature. Moreover, we also know that the bound is achieved by the extremal sources for (π, ρ, σ) envelope, and hence it is unlikely that we can do much better without more information about the moment generating function of the sources.

These results thus complement those reported in [19] in that we have results, from which we can compute bounds for both the mean buffer occupancy as well as their asymptotics. It is interesting to note that the extremal sources for both types of results have the same behavior. The former are of use in network design when the traffic is best effort while the results reported in this paper are better for tight QoS constraints [20].

It is of interest to extend these results to different scheduling disciplines as well as to the case of networks. While the many sources asymptotics are readily extendable to different scheduling disciplines via the methods presented in this paper, the extension to networks poses a daunting task, in particular when determining the traffic envelope at the output of a buffer.

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