On the Capacity of Ad hoc Networks under Random Packet Losses

Vivek P. Mhatre, Catherine P. Rosenberg, Ravi R. Mazumdar

***Abstract*— We consider the problem of determining asymptotic bounds on the capacity of a random ad hoc network. Previous approaches assumed a link layer model in which if a transmitter-receiver pair can communicate with each other, i.e., the Signal to Interference and Noise Ratio (SINR) is above a certain threshold, then the transmitted packet is received error-free by the receiver thereby. Using this model, the per node capacity of the network was shown to be $\Theta\left(\frac{1}{\sqrt{n\log n}}\right)$. In reality, for any finite link SINR, there is a non-zero probability of erroneous reception of the packet. We show that in a large network, as the packet travels an asymptotically large number of hops from source to destination, the cumulative impact of packet losses over intermediate links results in a per-node throughput of only $O\left(\frac{1}{n}\right)$ under the previously proposed routing and scheduling strategy. We then propose a new scheduling scheme to counter this effect. The proposed scheme provides tight guarantees on end-to-end packet loss probability, and improves the per-node throughput to $\Omega\left(\frac{1}{\sqrt{n}(\log n)^{\frac{\alpha+2}{2(\alpha-2)}}}\right)$ where $\alpha > 2$ is the path loss exponent.**

Keywords

Ad hoc networks, Capacity, Interference, Scheduling, SINR

I. Introduction

The problem of the capacity of wireless ad hoc networks was first analyzed by Gupta and Kumar in their seminal work [1]. This work was followed by several studies on the capacity of wireless ad hoc networks [3], [4], among others. The authors in [1] derive asymptotic bounds on the capacity of a random ad hoc network in which nodes are deployed randomly and uniformly over the surface of a sphere of unit area. Each node picks a random node as its destination node, and sends packets to that node by using multi-hop communication. All the nodes use a common transmit power level. The authors show that each node can achieve a throughput of $\Theta\left(\frac{1}{\sqrt{n\log n}}\right)$ packets per second. They also provide a scheduling and routing strategy that achieves this throughput.

The authors in [1] assume a link layer model in which, if the Signal to Interference and Noise Ratio (SINR) at the receiver is greater than a certain threshold then the packet is received successfully by the receiver. In practical wireless networks, for a given modulation and coding scheme, and for a fixed block length, as long as there is some noise and interference, i.e., as long as the SINR of a link is finite, there is a certain non-zero probability of packet error. Hence, the hypothesis of perfect packet reception when the link SINR is above a certain threshold can be realized only when one of the following is assumed: (i) infinite block length, or (ii) infinite number of retransmissions, i.e., retransmit the packet until it gets through.

Vivek P. Mhatre is with Motorola Inc., Arlington Heights, IL, USA. Email: mhatre@gmail.com, Catherine P. Rosenberg and Ravi R. Mazumdar are with the Department of Electrical and Computer Engineering, University of Waterloo, Canada. Email: {cath,mazum}@ece.uwaterloo.ca.

Both these approaches lead to unbounded delays, and hence under either (i) or (ii), a delay-capacity tradeoff study should be undertaken instead of a capacity study. In practical systems in order to keep the overall delay under tabs, the block length, as well as the maximum number of retransmission attempts (in case retransmissions are employed) are pre-determined, and fixed. Hence, while the threshold-based packet reception model used in [1] might be a reasonable choice for successful packet reception in a single hop network such as a cellular network, we argue that it needs to be refined when applied to a multi-hop network. In an ad hoc network, each packet traverses *multiple hops*. When a packet is relayed over a large number of links, each of which being likely to drop the packet with a certain probability (no matter how small it is), the end-to-end throughput can degrade significantly due to the *cumulative* packet error probability.

More generally, in this paper, we show that when studying capacity scaling problems, the underlying hypotheses have a significant impact on the results. In particular, we show that when packet losses over links are taken into account, we get strikingly different results thereby showing the sensitivity of capacity results to the underlying hypotheses.

For a large random ad hoc network, for a broad range of routing and scheduling schemes including the one proposed in [1], we show that the cumulative impact of per link packet loss results in a much lower per-node throughput of $O\left(\frac{1}{n}\right)$ instead of $\Theta\left(\frac{1}{\sqrt{n\log n}}\right)$. In order to counter the above throughput reduction due to cumulative packet error effect, we propose a new scheduling policy that uses lower spatial reuse to reduce interference, and thereby improves the SINR of each link. The proposed scheduling policy improves the end-to-end packet success probability, and results in a per-node throughput of $\Omega\left(\frac{1}{\sqrt{n}(\log n)^{\frac{\alpha+2}{2(\alpha-2)}}}\right)$, where $\alpha > 2$ is the path loss exponent.

The models and assumptions used in our analysis are presented in Section II. Section III contains results on throughput reduction due to cumulative packet loss. Our new scheduling policy, and the corresponding capacity results are presented in Section IV. Finally, we conclude in Section V.

II. Model for Random Networks

We use the standard ordering notations $O(.), o(.), \Omega(.), \omega(.), \Theta(.)$.

Scaling model: We assume that n nodes are randomly and uniformly deployed over a sphere S^2 of area n so that the node density is kept constant. Under this model, the far-field assumption can be employed in the path loss model [5]. The nodes are assumed to be stationary. Each node picks a random node as its destination node, and sends packets to this destination node. Each node uses the same transmit power P, and uses intermediate nodes as relays to reach its destination. We assume that as n scales, the modulation and coding scheme, as well as the block length remain fixed. Furthermore, we assume that a common frequency band of W units is available to all the nodes. This models the scenario in which although the network size increases, the underlying hardware and software capabilities of the nodes, as well as the available spectrum are unchanged.

Physical layer model: The SINR at a receiver node j when node i is transmitting a packet to j is as follows.

$$\text{SINR} = \frac{\frac{P}{|X_i - X_j|^{\alpha}}}{N + \sum_{k \in T,\ k \neq i} \frac{P}{|X_k - X_j|^{\alpha}}}, \tag{1}$$

where N is the noise power, X_i is the position vector of node i, T is the set of all the nodes that are transmitting simultaneously with node i, and $\alpha > 2$ is the path loss exponent. As in [1], we assume that the transmit power of all the nodes can be scaled with n so that the effects of receiver noise can be ignored, and the network becomes interference dominated. We therefore ignore the term N in the denominator of SINR henceforth. Two nodes can communicate with each other if the SINR at the receiver node is greater than a certain threshold, say β. In [1] the authors assume that if two nodes can communicate with each other, then the transmitted packet is received error-free. Thus, if $\hat{\phi}(\cdot)$ is the mapping between the SINR and the probability of successful packet reception, then the model used in [1] is equivalent to the following.

$$\hat{\phi}(\text{SINR}) = \begin{cases} 1 & \text{if SINR} \geq \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The key contribution of our work is to derive capacity results under the following more accurate physical layer model. We assume that the probability of a error-free packet reception is a continuous increasing function of SINR, $\phi(\cdot)$, that approaches unity as the SINR approaches infinity. Hence, for every finite SINR value, there is a non-zero probability that the packet is received in error which amounts to packet loss. The results carry over to the case in which the number of retransmissions allowed is finite but fixed, i.e., do not depend on n.

Interference model: We assume that the interference observed by a packet over different hops along its path from source to destination is independent across all the hops. Note that the set of interferers active during the transmission of a packet over different hops, and the actual packets that the interfering nodes transmit during those slots are unlikely to be correlated as the packet moves along its path. Furthermore, the interfering signals themselves, i.e., the sequences of bits in the interfering packets can be assumed to be independent of each other. Hence the assumption of independence of interference across multiple hops.

Network connectivity and routing: In order to ensure network connectivity, we do the following. The surface of the sphere is covered by a Voronoi tessellation in such a way that each Voronoi cell V can be enclosed inside a circle of radius $2\rho_n$, and each circle encloses a circle of radius ρ_n (all the distances are measured along the surface of S^2). Using a simple extension of connectivity results in [2], we show that if $\rho_n = \Theta\left(\sqrt{\log n}\right)$, then network connectivity can be ensured with high probability (see [6] for details). In particular, we choose ρ_n to be the radius of a circle of area $100 \log n$ on S^2. We assume the straight line routing policy from [1] in which packets are routed along straight line paths between source-destination pairs, i.e., every cell that intersects the straight line joining a source-destination pair, relays the packets of that pair.

Scheduling Policy: We first consider the scheduling policy proposed in [1], and obtain capacity results under this policy in Section III. We refer to this policy as π_1. This scheduling policy uses a vertex-coloring algorithm to guarantee that each cell gets a transmission opportunity at least once every K slots, where K is a constant that is independent of n. In other words, the length of the schedule is bounded even as n scales. In Section IV, we then propose a new improved scheduling policy, π_2 that yields better capacity under our physical layer model. The length of the schedule under π_2, K_n grows as a function of n,

III. Capacity under Scheduling Policy π_1 and continuous $\phi(\cdot)$

Recall that we assume that $\phi(.)$ approaches unity continuously as the SINR goes to infinity. Under this model, we show that the scheduling policy π_1 used in [1] results in a network capacity of $O\left(\frac{1}{n}\right)$.

Let L_i be the line segment along the surface of S^2 that connects the i-th source-destination pair (henceforth referred to as the i-th connection). We also use L_i to denote the length of the line segment joining the i-th source-destination pair. As per the straight line routing scheme, the packets of the i-ith connection are relayed hop-by-hop by every cell which intersects line L_i. Over each hop, any node in the relaying cell may forward the packet. The scheduling algorithm, π_1, and the uniform cell sizes ensure that communication between any two nodes in the neighboring cells is possible by guaranteeing that the SINR at the receiving node is greater than or equal to β (see [1] for more details). The following lemmas are presented without proofs (see [6] for proofs).

Lemma 1: For the straight line routing scheme, the number of hops H_i for connection i is $\Theta\left(\frac{L_i}{\rho_n}\right)$. More precisely,

$$\frac{1}{8}\frac{L_i}{\rho_n} \leq H_i \leq \frac{16}{\pi}\frac{L_i}{\rho_n}.$$

Lemma 2: Fix t such that $0 < t < 1$. For connection i, out of H_i total hops, let h_i hops be such that each of these hops covers a distance of less than $t\rho_n$. Then

$$H_i - h_i \geq \frac{L_i}{\rho_n}\left(\frac{1 - \frac{16t}{\pi}}{8 - t}\right).$$

Thus, for the above $H_i - h_i$ hops, the signal received at the receiver is at the most $P(t\rho_n)^{-\alpha}$.

In [1], scheduling policy π_1 was proposed to guarantee an SINR of at least β at the receiver of every scheduled transmission. This scheduling policy corresponds to a graph coloring problem, and it was shown that the maximum number of colors required to color all the cells is upper bounded by $1+c_1$, where c_1 is a fixed constant that is independent of n (see Lemma 4.4 in [1]). Using this scheduling scheme, each cell gets a transmission opportunity at least once every $1+c_1$ time slots. The following lemma shows that under such a scheduling strategy, except for a small fraction of hops, all the remaining hops of a connection receive a certain minimum amount of interference from other simultaneous transmissions.

Lemma 3: Fix $M > 9$. Let N_i be the number of hops of connection i such that there is no simultaneous interfering transmission within a circle of radius $(M+8)\rho_n$ around the receivers of those hops. Then,

$$N_i \leq \frac{L_i}{\rho_n}\left(\frac{2(1+c_1)}{M}\right),$$

where c_1 is the constant from Lemma 4.4 in [1].

Using the above lemma, we have the following important result.

Proposition 1: There exist fixed constants t_0 and M_0, that do not depend on n, such that for at least $L_i/16\rho_n$ hops of connection i, the SINR is less than a fixed constant β_0 where

$$\beta_0 = \left(\frac{M_0+8}{t_0}\right)^{\alpha}.$$

Since $\phi(\cdot)$ is continuous, and the SINR is upper bounded by a fixed constant β_0, the probability of successful packet reception for these hops is also upper bounded by a fixed constant $\phi(\beta_0) < 1$.

Proof: Let A_i be the set of hops of connection i for which the received signal is at the most $P(t\rho_n)^{-\alpha}$. Then, given $\epsilon_1 > 0$, we can find $t > 0$ small enough so that using Lemma 2,

$$|A_i| \geq \frac{L_i}{\rho_n}\left(\frac{1}{8}-\epsilon_1\right). \tag{3}$$

Let B_i be the set of hops of connection i for which there is no simultaneous transmissions within a distance of $(M+8)\rho_n$ of the receiver. Using Lemma 3, $|B_i| = N_i$, and given $\epsilon_2 > 0$, we can choose M large enough so that

$$|B_i| \leq \frac{L_i}{\rho_n}\left(\frac{2(1+c_1)}{M}\right) \leq \frac{L_i}{\rho_n}\epsilon_2. \tag{4}$$

Thus using (3) and (4),

$$\begin{aligned}|A_i \cap {B_i}^c| &\geq |A_i| - |B_i| \\ &= \frac{L_i}{\rho_n}\left(\frac{1}{8}-\epsilon_1-\epsilon_2\right).\end{aligned}$$

If we pick $\epsilon_1 = \epsilon_2 = 1/32$, and choose $t = t_0$ and $M = M_0$ corresponding to this choice of ϵ_1, ϵ_2, then

$$|A_i \cap {B_i}^c| \geq \frac{L_i}{16\rho_n}. \tag{5}$$

Note that $A_i \cap {B_i}^c$ is the set of hops over which the received signal is no more than $P(t_0\rho_n)^{-\alpha}$, *and* there is at least one simultaneous transmission within a distance of $(M_0+8)\rho_n$ of the receiver. This in turn means that for these hops, the SINR is upper bounded by

$$SINR \leq \frac{P(t_0\rho_n)^{-\alpha}}{P\left((M_0+8)\rho_n\right)^{-\alpha}} = \left(\frac{M_0+8}{t_0}\right)^{\alpha} = \beta_0. \tag{6}$$

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As the number of hops between source and destination scales to infinity, and as per the above result, a fixed fraction of those hops have a certain minimum non-zero packet loss probability. Hence, we can show that the expected end-to-end throughput of a node after accounting for the packet losses is given by the following result.

Proposition 2: Under scheduling policy π_1, continuous $\phi(\cdot)$, and with ρ_n chosen to be the radius of a disk of area $100\log n$, the per-node throughput that can be achieved is $O(\frac{1}{n})$.

Outline of the proof: The proof consists of two parts. In the first part, we show that under scheduling policy π_1, the maximum rate, λ_n, at which a node can inject packets into the network is upper bounded by $\frac{1}{50\log n}$. This result follows from the fact that each cell contains at least $50\log n$ nodes with high probability. In the second part of the proof, we show that the end-to-end probability of packet delivery for any connection decays as $\frac{\log n}{n}$. These two results imply that when intermediate packet losses are accounted for, the destination node will receive packets at a rate of $O\left(\frac{1}{n}\right)$. In the following, we provide details of the second part of the proof.

We know from Proposition 1 that, among the H_i hops of connection i, at least $L_i/16\rho_n$ hops have a probability of packet success of no more than $\phi(\beta_0)$. Thus,

$$\Lambda_n \leq \lambda_n \left\{\phi(\beta_0)\right\}^{\frac{L_i}{16\rho_n}}. \tag{7}$$

Note that L_i are i.i.d. random variables. By taking the expectation with respect to L_i, the end-to-end throughput $\mathbf{E}[\Lambda_n]$ is,

$$\begin{aligned}\mathbf{E}[\Lambda_n] = \mathbf{E}_L[\Lambda_n] &\leq \lambda_n\, \mathbf{E}_L\left\{\left(\left\{\phi(\beta_0)\right\}^{\frac{1}{16\rho_n}}\right)^{L_i}\right\} \\ &= \lambda_n\, \mathbf{E}_L\left[\delta^{L_i}\right].\end{aligned} \tag{8}$$

where we have substituted $\delta = \left\{\phi(\beta_0)\right\}^{\frac{1}{16\rho_n}}$. Note that in determining the average end-to-end throughput $\mathbf{E}[\Lambda_n]$, we take expectations at two levels; once to take into account the randomness due to the possibility of packet error on each link, and once to take into account the randomness due to the locations of the source and destination nodes. Also note that $0 < \delta < 1$. Since L_i is a line connecting two points picked at random on the surface of S^2, we can show that (see [6] for proof).

$$\mathbf{E}_L\left[\delta^{L_i}\right] = \frac{2\pi\left(1+\delta^{\frac{\sqrt{\pi n}}{2}}\right)}{4\pi + n(\log\delta)^2}. \tag{9}$$

Using (8) and (9), and substituting $\delta = \left\{\phi(\beta_0)\right\}^{\frac{1}{16\rho_n}}$, we get

$$\begin{aligned}\mathbf{E}[\Lambda_n] &\leq \lambda_n \frac{512\pi{\rho_n}^2\left(1+\left\{\phi(\beta_0)\right\}^{\frac{\sqrt{\pi n}}{32\rho_n}}\right)}{(1024\pi{\rho_n}^2 + n(\log\phi(\beta_0))^2)} \\ &< \lambda_n \frac{1024\pi{\rho_n}^2}{(1024\pi{\rho_n}^2 + n(\log\phi(\beta_0))^2)}, \text{ since } \phi(\beta_0) < 1. \\ &< \lambda_n \frac{1024\pi{\rho_n}^2}{n(\log\phi(\beta_0))^2} \\ &\leq \lambda_n \frac{c_0\log n}{n}, \text{ since } \rho_n = \Theta\left(\sqrt{\log n}\right) \\ &\leq \frac{1}{50\log n}\frac{c_0\log n}{n} = O\left(\frac{1}{n}\right)\end{aligned}$$

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Remark: The above bound on the throughput also holds for a broad class of routing schemes, i.e., Proposition 2 holds even when the assumption of straight-line routing is relaxed, (see [6] for details).

IV. Capacity under a new Scheduling Policy π_2, and continuous $\phi(\cdot)$

In this section, we show that if we reduce the extent of spatial reuse via scheduling, then we can bound the end-to-end packet loss probability, *not just* the per link packet loss probability. This restricts the throughput reduction due to cumulative packet loss. The proofs for the following lemmas can be found in [6].

Lemma 4: For a given $\epsilon > 0$, if a scheduling policy ensures that each transmission has an SINR of at least $\beta_n \geq \log n$, and if the network is large enough, i.e., $n > \frac{4}{5\sqrt{\pi}\epsilon}$, then each connection has an end-to-end cumulative packet loss probability of no more than ϵ.

Lemma 5: For each scheduled transmission of a cell an SINR of at least $\beta_n = \log n$ is guaranteed if all the cells that are within a distance R_n of the given cell are silent, where R_n is given by:

$$R_n = 20\sqrt{\log n}\left(+4\left(\frac{256\log n}{(\alpha-2)}\right)^{\frac{1}{(\alpha-2)}}\right). \tag{10}$$

Concurrent transmissions that are at least R_n distance apart can then be scheduled using the following policy.

Scheduling policy π_2: Consider the graph in which each cell is represented by a vertex. Two vertices of the graph have an edge between them if and only if their corresponding cells are within a distance of R_n from each other. Since the area of each cell is lower bounded by $50\pi\log n$, the number of cells within a distance R_n of a cell, and correspondingly the maximum degree of a vertex in the corresponding graph is upper bounded by V_n given by

$$V_n = 8\left(1+4\left(\frac{256\log n}{(\alpha-2)}\right)^{\frac{1}{\alpha-2}}\right)^2. \tag{11}$$

Since a graph in which the maximum degree of a node is k can be vertex colored using no more than $k+1$ colors in polynomial time [8], we use this algorithm to vertex color the graph. Thus, the required number of colors is no more than $K_n = V_n + 1$, which is a function of n unlike π_1 where the number of colors is a constant independent of n.

Under scheduling policy π_2 we show that the received interference power is bounded [6]. Indeed it can be seen that for a path loss exponent $\alpha > 2$ all r-th moments of the interference for $r \geq 2$ are bounded. Moreover, since all moments exist, in [6], we show that the interference satisfies the large deviations principle, i.e., a moment generating function of the interference is well defined, and hence the distribution function of interference has exponentially decaying tail.

Proposition 3: For a given fixed $\epsilon > 0$ with scheduling policy π_2, for a large network, i.e., if $n > \frac{4}{5\sqrt{\pi}\epsilon}$, each connection has an end-to-end packet loss probability of less than ϵ.

Proof: The proof follows from exponential tail decay, see [6]. ∎

We thus have the following main result.

Proposition 4: When random packet losses over intermediate links are taken into account, the per-node throughput in a large ad hoc network is $\Omega\left(\frac{1}{\sqrt{n}(\log n)^{\frac{\alpha+2}{2(\alpha-2)}}}\right)$, and the bound can be achieved under Scheduling Policy π_2.

Proof: If each connection generates packets at a rate of λ_n, then from Lemma 4.12 in [1], the total load of transmitting packets on each cell is upper bounded by $c_5\lambda_n\sqrt{n\log n}$ with high probability. When the cell gets an opportunity to transmit, it uses the entire bandwidth to transmit at a rate of W. Thus, with high probability, a rate of λ_n can be scheduled if the following holds.

$$c_5\lambda_n\sqrt{n\log n} \leq \frac{W}{K_n}$$
$$\Rightarrow \lambda_n \leq \frac{W}{c_5\sqrt{n\log n}\left(a_0 + a_1(\log n)^{\frac{1}{(\alpha-2)}} + a_2(\log n)^{\frac{2}{\alpha-2}}\right)} \tag{12}$$

using $K_n = V_n + 1$, and (11), and where a_0, a_1, and a_2 are constants that do not depend on n.

The goodput after accounting for cumulative packet losses is then given by:

$$\mathbf{E}[\Lambda_n] = \frac{(1-\epsilon)W}{c_5\sqrt{n\log n}\left(a_0 + a_1(\log n)^{\frac{1}{(\alpha-2)}} + a_2(\log n)^{\frac{2}{\alpha-2}}\right)} = \Omega\left(\frac{1}{\sqrt{n}(\log n)^{\frac{\alpha+2}{2(\alpha-2)}}}\right).$$

∎

V. Conclusion

Our results show that idealized modeling assumptions can lead to optimistic conclusions, and hence for scaling techniques to provide useful guidelines it is important to understand the limitations of the underlying assumptions.

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