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Variance expressions for spectra estimated using auto-regressions☆

Liang-Liang Xie[a], Lennart Ljung[b,*]

[a] *Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China*
[b] *Department of Electrical Engineering, Linköping University, Linköping S-581 83, Sweden*

Abstract

An expression for the variance of the estimated spectrum based on auto-regressions is developed. This expression is asymptotic in the number of data, but exact in the model order. As the order tends to infinity it converges to the well known result that the variance is proportional to the model order times the square of the spectrum itself. The exact expression gives insight into the character of this convergence, its speed and its dependence on the poles of the underlying AR-process.

Keywords: Variance; Spectra; Auto-regression

1. Introduction

In this paper we will study the asymptotic properties of spectral estimates based on Autoregressive (AR) models of time series. This is of course a much-studied subject with many well known results. Our contribution will be an exact expression for the variance of the asymptotic distribution of the spectral estimate for finite order AR models. In particular, this result illuminates the character of the convergence to the well known asymptotic expressions as the order of the AR model tends to infinity. The role of the poles of the underlying true AR-process will also be displayed.

The setup is as follows: Consider a stationary time series $y(t), t = 1, 2, \ldots$. Denote its spectral density by $\Phi_y(\omega)$. From N observations from the time series, an AR-model of order n is estimated by minimizing

$$\sum_{t=n+1}^{N} (y(t) + a_1 y(t-1) + \cdots + a_n y(t-n))^2, \tag{1}$$

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* Corresponding author. Tel.: +46-1328-1310; fax: +46-1328-2622.
E-mail address: ljung@isy.liu.se (L. Ljung).

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w.r.t. $a_k, 1 \leqslant k \leqslant n$, yielding the estimates $\hat{a}_k(N)$. The variance of the innovation process is estimated by

$$\hat{\lambda}_N = \frac{1}{N-n} \sum_{t=n+1}^{N} (y(t) + \hat{a}_1(N)y(t-1) + \cdots + \hat{a}_n(N)y(t-n))^2. \tag{2a}$$

Then the estimate of the spectral density of y is formed as

$$\hat{\Phi}_N(\omega) = \frac{\hat{\lambda}_N}{|\hat{A}_N(\omega)|^2}, \tag{2b}$$

$$\hat{A}_N(\omega) = 1 + \hat{a}_1(N)\mathrm{e}^{-\mathrm{j}\omega} + \cdots + \hat{a}_n(N)\mathrm{e}^{-\mathrm{j}n\omega}. \tag{2c}$$

The asymptotic properties of this estimate are well known, e.g. Hannan (1970) or Hannan and Deistler (1988). Suppose that the given process $y(t)$ indeed can be described by a rth order AR-process:

$$y(t) + a_1 y(t-1) + \cdots + a_r y(t-r) = e(t), \tag{3a}$$

$$\mathrm{E}e^2(t) = \lambda, \tag{3b}$$

$$\mathrm{E}e^3(t) = 0, \tag{3c}$$

$$\mathrm{E}(e^2(t) - \lambda)^2 = \mu, \tag{3d}$$

where $e(t)$ is a sequence of i.i.d. random variables. (The asymptotic properties below also hold under weaker assumptions on e.) Then, if $n \geqslant r$ and we denote

$$\hat{\theta}_N = \begin{pmatrix} \hat{a}_1(N) \\ \vdots \\ \hat{a}_n(N) \end{pmatrix} \tag{4}$$

and let θ_0 be the corresponding vector of the true AR-parameters, we have that

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in \mathrm{AsN}(0, P), \tag{5a}$$

$$P = \lambda[\mathrm{E}\psi_t\psi_t^{\mathrm{T}}]^{-1}, \tag{5b}$$

$$\psi_t = \begin{pmatrix} y(t-1) \\ \vdots \\ y(t-n) \end{pmatrix}, \tag{5c}$$

$$\sqrt{N}(\hat{\lambda}_N - \lambda) \in \mathrm{AsN}(0, \mu). \tag{5d}$$

Here $x_N \in \mathrm{AsN}(a, B)$ means that the random variable x_N converges in distribution to the normal distribution with mean a and covariance matrix B as N tends to infinity.

Moreover, the asymptotic joint distribution of the two estimates is normal with $\hat{\theta}$ independent of $\hat{\lambda}$.

From (5) all results about the asymptotic distribution of $\hat{\Phi}_N(\omega)$ readily can be derived. We will be particularly interested in the variance of the asymptotic distribution of $\sqrt{N}(\hat{\Phi}_N(\omega) - \Phi_y(\omega))$ and denote it by $\Lambda_n(\omega)$:

$$\sqrt{N}(\hat{\Phi}_N(\omega) - \Phi_y(\omega)) \in \mathrm{AsN}(0, \Lambda_n(\omega)). \tag{6}$$

It can be computed from (5) using e.g. Lemma 2.1 below. As the order of the AR-model, n, tends to infinity, the expression for $\Lambda_n(\omega)$ simplifies considerably:

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_n(\omega) = \begin{cases} 2\Phi_y^2(\omega) & \text{if } \omega \neq 0, \omega \neq \pi, \\ 4\Phi_y^2(\omega) & \text{if } \omega = 0 \text{ or } \omega = \pi. \end{cases} \tag{7}$$

This has been proved in Kromer (1970), for the case of a fixed regression order, and in Berk (1974) for the case where n tends to infinity and $n^3/N \to 0$ as $N \to \infty$. The case of an input (exogenous process) present is treated in e.g., Ljung and Wahlberg (1992) and Hannan and Kavalieris (1984).

The purpose of the current paper is to give an explicit expression for $\Lambda_n(\omega)$ as a function of n. An important observation, established in Ninness et al. (1999b), is that the calculation to achieve this is made much simpler by choosing another orthonormal basis. In our case this basis should be constructed with the poles of the true underlying AR-process. The explicit expression will display the nature of the convergence taking place in (7), the convergence rate, and also the role that is played by the poles of the true underlying AR-process. Related results for processes with an exogenous input are described in Ninness et al. (1999a), Xie and Ljung (2001) and Ninness and Hjalmarsson (2002a, b).

2. A preliminary result

We will use polynomials in the shift operator q for efficient notation and write

$$A_0(q)y(t) = e(t) \tag{8}$$

for the true AR-representation (3a). This means that

$$A_0(q) = 1 + a_1 q^{-1} + \cdots + a_r q^{-r}.$$

Similarly the nth order model used in (1)–(2) will be written as

$$A(q, \theta)y(t) = e(t), \tag{9}$$

where θ collects the parameters $a_k, 1 \leqslant k \leqslant n$ in a column vector. The corresponding estimate is denoted $\hat{\theta}_N$ as in (4).

Thus, from (2c)

$$\hat{A}_N(\omega) = A(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) = 1 + \sum_{k=1}^{n} \hat{a}_k(N)\mathrm{e}^{-\mathrm{j}\omega k} = 1 + W^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega})\hat{\theta}_N, \tag{10}$$

where $\hat{\theta}_N$ contains the LS-estimated AR-parameters and with

$$W(e^{j\omega}) \triangleq [e^{-j\omega}, e^{-2j\omega}, \ldots, e^{-nj\omega}]^T.$$

To proceed, we need several times a standard result about convergence in distribution. For a proof, see, e.g., Rao (1973, Theorem 6a.2(ii), p. 387).

Lemma 2.1. *Let $\hat{X}_N$ be a sequence of k-dimensional estimates of parameter vector X_0 such that the asymptotic distribution of $\sqrt{N}[\hat{X}_N - X_0]$ is k-variate normal with mean zero and covariance matrix Σ. Further let g be a function of k variables which is differentiable in a neighborhood of X_0. Then the asymptotic distribution of $\sqrt{N}(g(\hat{X}_N) - g(X_0))$ is normal with mean zero and variance*

$$v(X_0) = J^T \Sigma J, \quad \text{with } J = \left.\frac{\partial g}{\partial X}\right|_{X=X_0},$$

provided $v(X_0) \neq 0$.

Using this lemma we first prove a preliminary result about the asymptotic distribution of the model polynomial (10):

Theorem 2.1. *Consider the* AR-*process* (8) *of order r. Let the A-polynomial be estimated as an n:th order* AR *model* (9) *and assume that $n \geqslant r$. Let the roots of the true* AR-*polynomial $z^r A_0(z)$ be α_k, $k = 1, \ldots, r$. Then the real part and the imaginary part of the $A(e^{j\omega}, \hat{\theta}_N)$-function are both asymptotically distributed as*

$$\sqrt{N}(\mathrm{Re}\{A(e^{j\omega}, \hat{\theta}_N)\} - \mathrm{Re}\{A_0(e^{j\omega})\}) \in \mathrm{AsN}(0, v_1),$$

$$\sqrt{N}(\mathrm{Im}\{A(e^{j\omega}, \hat{\theta}_N)\} - \mathrm{Im}\{A_0(e^{j\omega})\}) \in \mathrm{AsN}(0, v_2),$$

$$\text{with } v_1 + v_2 = |A_0(e^{j\omega})|^2 \left[(n - r) + \sum_{m=1}^{r} \frac{1 - |\alpha_m|^2}{|e^{j\omega} - \alpha_m|^2}\right]. \tag{11}$$

Proof. From (5) the asymptotic distribution of the LS-estimated AR-parameters $\hat{\theta}_N$ is given by

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in \mathrm{AsN}(0, \lambda R^{-1}), \tag{12}$$

where

$$R = \mathrm{E}\{\psi_t \psi_t^T\}, \quad \psi_t \triangleq \frac{d}{d\theta}[A(q, \theta) y(t)] = [y(t-1), y(t-2), \ldots, y(t-n)]^T. \tag{13}$$

By Parseval's relation and Eq. (8),

$$R = T(\Phi_y(\omega)), \quad \Phi_y(\omega) = \frac{\lambda}{|A_0(e^{j\omega})|^2}, \tag{14}$$

where

$$T(f(\omega)) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\omega}) W^*(e^{j\omega}) f(\omega)\, d\omega, \tag{15}$$

with $W(e^{j\omega})$ defined in (10) and $(\cdot)^*$ denoting the conjugate transpose.

Then by Lemma 2.1 and (10),

$$\sqrt{N}(\mathrm{Re}\{A(\mathrm{e}^{\mathrm{j}\omega},\hat{\theta}_N)\} - \mathrm{Re}\{A_0(\mathrm{e}^{\mathrm{j}\omega})\}) \in \mathrm{AsN}(0, v_1),$$

$$\text{with } v_1 = \lambda \mathrm{Re}\{W^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega})\} T^{-1}(\Phi_y(\omega))\,\mathrm{Re}\{W(\mathrm{e}^{\mathrm{j}\omega})\};$$

$$\sqrt{N}(\mathrm{Im}\{A(\mathrm{e}^{\mathrm{j}\omega},\hat{\theta}_N)\} - \mathrm{Im}\{A_0(\mathrm{e}^{\mathrm{j}\omega})\}) \in \mathrm{AsN}(0, v_2),$$

$$\text{with } v_2 = \lambda\, \mathrm{Im}\{W^{\mathrm{T}}(\mathrm{e}^{\mathrm{j}\omega})\} T^{-1}(\Phi_y(\omega))\,\mathrm{Im}\{W(\mathrm{e}^{\mathrm{j}\omega})\}.$$

Then it is not difficult to check that

$$v_1 + v_2 = \lambda W^*(\mathrm{e}^{\mathrm{j}\omega}) T^{-1}(\Phi_y(\omega)) W(\mathrm{e}^{\mathrm{j}\omega}). \tag{16}$$

But it is hard to directly convert (16) into a simple analytic form, which obviously should only depend on A_0.

In order to get a simple analytic expression, we introduce a virtual time series $\{v(t)\}$ defined by

$$A_1(q)y(t) = v(t). \tag{17}$$

Then by (9), we have a new model

$$\frac{A(q,\theta)}{A_1(q)}\, v(t) = e(t). \tag{18}$$

The model (18) may not be actually used, but using it will result in the same variance as using (9), and the expression may be easier to simplify. That is indeed the case if $A_1 = A_0$. Then $v(t) = e(t)$ is i.i.d. and the new model (18) becomes

$$\frac{A(q,\theta)}{A_0(q)}\, v(t) = e(t). \tag{19}$$

Now, supposing $\alpha_k, 1 \leqslant k \leqslant r$ are the roots of $A_0(q)$ ($\max_{1\leqslant k\leqslant r}|\alpha_k| < 1$ for the AR-process (8) to be stationary), if we reparameterize model (19) using the orthonormal basis (see Ninness et al., 1999b)

$$\Gamma_{n,r}(q) = [\gamma_1(q), \gamma_2(q), \ldots, \gamma_n(q)]^{\mathrm{T}}, \tag{20}$$

where, $\gamma_i(q), 1 \leqslant i \leqslant n$ is defined by

$$\gamma_i(q) \triangleq \begin{cases} q^{-i}, & 1 \leqslant i \leqslant n-r, \\ b_{i-(n-r)}(q)\, q^{-(n-r)}, & n-r < i \leqslant n, \end{cases} \tag{21}$$

with

$$b_k(q) \triangleq \frac{\sqrt{1-|\alpha_k|^2}}{q-\alpha_k} \prod_{m=1}^{k-1} \frac{1-\bar{\alpha}_m q}{q-\alpha_m}, \qquad 1 \leqslant k \leqslant r, \tag{22}$$

then we have a new equivalent model

$$A'(q,\theta')v(t) = e(t), \tag{23}$$

$$\begin{aligned} \text{with } A'(q,\theta') &\triangleq \frac{1}{A_0(q)} + (\theta')^* \cdot \Gamma_{n,r}(q) \\ &= \frac{1}{A_0(q)} + \bar{a}'_1 q^{-1} + \cdots + \bar{a}'_{n-r} q^{-(n-r)} \\ &\quad + \bar{a}'_{n-r+1} b_1(q) q^{-(n-r)} + \cdots + \bar{a}'_n b_r(q) q^{-(n-r)}, \end{aligned} \tag{24}$$

where $\theta' \triangleq [a'_1, a'_2, \ldots, a'_n]^{\mathrm{T}}$ are the new parameters and $(\cdot)^*$ denotes the conjugate transpose. Note that θ' would be complex if A_0 has complex roots. Actually it is easy to check that there exist linear transformations between the old and new parameters, and between the old and new orthonormal basis:

$$\theta = \Pi \cdot \theta', \tag{25}$$

$$\Gamma_{n,r}(q) = \Pi^* \cdot \frac{W(q)}{A_0(q)}, \tag{26}$$

where, $W(q) = [q^{-1}, q^{-2}, \ldots, q^{-n}]^{\mathrm{T}}$, and Π is an $n \times n$ constant matrix, whose entries only depend on the roots of A_0: $\alpha_k, 1 \leqslant k \leqslant r$.

Let

$$\hat{\theta}'_N \triangleq \Pi^{-1} \cdot \hat{\theta}_N, \tag{27}$$

where $\hat{\theta}_N$ are the LS-estimates of the parameters θ in (4).

By (10), (24), (26)–(27) and noticing that $\hat{\theta}_N$ are real estimates, we have

$$\begin{aligned} A(\mathrm{e}^{\mathrm{j}\omega}, \hat{\theta}_N) &= 1 + \hat{\theta}_N^{\mathrm{T}} \cdot W(\mathrm{e}^{\mathrm{j}\omega}) = 1 + (\hat{\theta}'_N)^* \cdot \Pi^* \cdot \frac{W(\mathrm{e}^{\mathrm{j}\omega})}{A_0(\mathrm{e}^{\mathrm{j}\omega})} \cdot A_0(\mathrm{e}^{\mathrm{j}\omega}) \\ &= 1 + (\hat{\theta}'_N)^* \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega}) \\ &= 1 + (\hat{\theta}_N)^{\mathrm{T}} \cdot (\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega}). \end{aligned}$$

Hence, by Lemma 2.1 and (12), we have

$$v_1 = \lambda \operatorname{Re}\{(\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega})\}^{\mathrm{T}} R^{-1} \operatorname{Re}\{(\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega})\},$$

$$v_2 = \lambda \operatorname{Im}\{(\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega})\}^{\mathrm{T}} R^{-1} \operatorname{Im}\{(\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega})\}.$$

Therefore, by (13) and noticing that ψ_t is real,

$$\begin{aligned} v_1 + v_2 &= \lambda\{(\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega})\}^* R^{-1} \{(\Pi^*)^{-1} \cdot \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \cdot A_0(\mathrm{e}^{\mathrm{j}\omega})\} \\ &= \lambda |A_0(\mathrm{e}^{\mathrm{j}\omega})|^2 \Gamma_{n,r}^*(\mathrm{e}^{\mathrm{j}\omega}) (\mathrm{E}\{\Pi^* \psi_t \psi_t^* \Pi\})^{-1} \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}) \\ &= |A_0(\mathrm{e}^{\mathrm{j}\omega})|^2 \Gamma_{n,r}^*(\mathrm{e}^{\mathrm{j}\omega}) \Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega}), \end{aligned} \tag{28}$$

where, the last equality follows from (3b), (8), (26) and the orthonormality of $\Gamma_{n,r}(q)$ with

$$\Pi^* \psi_t = \Pi^* W(q) y(t) = \Gamma_{n,r}(q) A_0(q) y(t) = \Gamma_{n,r}(q) e(t). \quad (29)$$

Finally, by explicitly expressing (28) in terms of (20)–(22), we have (11). □

Although we aim at a result about the asymptotic distribution of the spectrum estimate (6), the result of Theorem 2.1 is of independent interest. It shows that when n tends to infinity, the variance of the estimated A-polynomial behaves like $\sim n|A_0(e^{j\omega})|^2$. It also shows how the poles α_m of the true AR-representation affect the variance for finite orders n.

3. The main result

Let us now turn to the main result, an expression for $\Lambda_n(\omega)$ in (6).

Theorem 3.1. *Consider the* AR-*process* (8) *of order* r. *Let the* A-*polynomial be estimated as an nth order* AR *model* (9) *and assume that* $n \geqslant r$. *Let the roots of the true* AR-*polynomial* $z^r A_0(z)$ *be* $\alpha_k, k = 1, \ldots, r$ *and arrange them in the following way: the* $r - 2\rho$ *first ones are real and the remaining* 2ρ *ones occur in complex conjugate pairs*: $\alpha_{r-2\rho+2k-1} = \beta_k$, $\alpha_{r-2\rho+2k} = \bar{\beta}_k, k = 1, \ldots, \rho$. *From all the roots define*

$$\Pi_k(\omega) \triangleq e^{-2(n-r)j\omega} \prod_{m=1}^{k} \frac{(1 - \bar{\alpha}_m e^{j\omega})^2}{(e^{j\omega} - \alpha_m)^2}, \quad 0 \leqslant k \leqslant r - 1, \quad (30)$$

and from the complex conjugated roots define for $1 \leqslant i \leqslant \rho$,

$$C(\beta_i) \triangleq (1 - |\beta_i|^2) \cdot \frac{(1 - \beta_i e^{j\omega})(1 - \bar{\beta}_i e^{j\omega}) + (e^{j\omega} - \beta_i)(e^{j\omega} - \bar{\beta}_i)}{(e^{j\omega} - \beta_i)^2 (e^{j\omega} - \bar{\beta}_i)^2}. \quad (31)$$

Then the estimated spectrum defined by (2) *is asymptotically distributed as*

$$\sqrt{N}(\hat{\Phi}_N(\omega) - \Phi_y(\omega)) \in \mathrm{AsN}(0, \Lambda_n(\omega)) \quad (32)$$

with

$$\Lambda_n(\omega) = \frac{\mu}{\lambda^2} \Phi_y^2(\omega) + 2\Phi_y^2(\omega) S_n(\omega, A_0), \quad (33)$$

where S_n *is defined by*

$$S_n(\omega, A_0) = (n - r) + \sum_{p=1}^{n-r} \cos(2p\omega) + \sum_{k=1}^{r} \frac{1 - |\alpha_k|^2}{|e^{j\omega} - \alpha_k|^2} + \mathrm{Re}\left\{ \sum_{k=1}^{r-2\rho} \frac{1 - |\alpha_k|^2}{(e^{j\omega} - \alpha_k)^2} \Pi_{k-1}(\omega) + \sum_{i=1}^{\rho} C(\beta_i) \Pi_{r-2(\rho-i+1)}(\omega) \right\}, \quad (34)$$

with $C(\beta_i)$, $\Pi_k(\omega)$ *defined in* (31) *and* (30).

Proof. We first establish that the squared amplitude of the $A(\mathrm{e}^{\mathrm{j}\omega},\hat\theta_N)$-function is asymptotically distributed as

$$\sqrt{N}(|A(\mathrm{e}^{\mathrm{j}\omega},\hat\theta_N)|^2-|A_0(\mathrm{e}^{\mathrm{j}\omega})|^2)\in \mathrm{AsN}(0,2|A_0(\mathrm{e}^{\mathrm{j}\omega})|^4 S_n(\omega,A_0)). \tag{35}$$

By Lemma 2.1 and (5a), we have (for simplicity, denote $\hat A_N := A(\mathrm{e}^{\mathrm{j}\omega},\hat\theta_N)$ and the argument $(\mathrm{e}^{\mathrm{j}\omega})$ is omitted in the following equations)

$$\sqrt{N}(|\hat A_N|^2-|A_0|^2)\in \mathrm{AsN}(0,v_3),$$

$$\text{with } v_3=[2\,\mathrm{Re}\,A_0\,\mathrm{Re}\,W+2\,\mathrm{Im}\,A_0\,\mathrm{Im}\,W]^{\mathrm{T}}P[2\,\mathrm{Re}\,A_0\,\mathrm{Re}\,W+2\,\mathrm{Im}\,A_0\,\mathrm{Im}\,W].$$

After some calculation, it is not difficult to verify that

$$v_3=2|A_0|^2W^*PW+2\,\mathrm{Re}\{\bar A_0^2W^{\mathrm{T}}PW\}. \tag{36}$$

Now, by (5), (26) and (29), we have

$$\begin{aligned} W^{\mathrm{T}}PW &= \lambda W^{\mathrm{T}}[\mathrm{E}\psi_t\psi_t^{\mathrm{T}}]^{-1}W\\ &=\lambda[A_0(\Pi^*)^{-1}\Gamma_{n,r}]^{\mathrm{T}}[\mathrm{E}\psi_t\psi_t^{\mathrm{T}}]^{-1}[A_0(\Pi^*)^{-1}\Gamma_{n,r}]\\ &=\lambda A_0^2\Gamma_{n,r}^{\mathrm{T}}[\mathrm{E}\Pi^*\psi_t(\Pi^*\psi_t)^{\mathrm{T}}]^{-1}\Gamma_{n,r}\\ &=\lambda A_0^2\Gamma_{n,r}^{\mathrm{T}}[\mathrm{E}\Gamma_{n,r}(q)e_t\Gamma_{n,r}^{\mathrm{T}}(q)e_t]^{-1}\Gamma_{n,r}. \end{aligned} \tag{37}$$

To continue on (37), by Parseval's relation and (3b), we have

$$\begin{aligned} R_1 &\triangleq \mathrm{E}\Gamma_{n,r}(q)e_t\Gamma_{n,r}^{\mathrm{T}}(q)e_t\\ &=\frac{\lambda}{2\pi}\int_{-\pi}^{\pi}\Gamma_{n,r}(\mathrm{e}^{\mathrm{j}\omega})\Gamma_{n,r}^{\mathrm{T}}(\mathrm{e}^{-\mathrm{j}\omega})\,\mathrm{d}\omega. \end{aligned}$$

Note that A_0 has ρ pairs of complex conjugate roots: $\alpha_{r-2\rho+1}=\beta_1$, $\alpha_{r-2\rho+2}=\bar\beta_1,\ldots,\alpha_{r-1}=\beta_\rho$, $\alpha_r=\bar\beta_\rho$. By (20)–(22), one finds that

$$R_1=\begin{bmatrix} I_{n-2\rho} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathscr{B}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathscr{B}_\rho \end{bmatrix}, \tag{38}$$

where for $i=1,\ldots,\rho$,

$$\mathscr{B}_i=\frac{1}{1-\beta_i^2}\begin{bmatrix} 1-|\beta_i|^2 & \beta_i-\bar\beta_i\\ \beta_i-\bar\beta_i & 1-|\beta_i|^2\end{bmatrix},$$

and

$$\mathscr{B}_i^{-1}=\frac{1}{1-\bar\beta_i^2}\begin{bmatrix} 1-|\beta_i|^2 & \bar\beta_i-\beta_i\\ \bar\beta_i-\beta_i & 1-|\beta_i|^2\end{bmatrix}.$$

Finally, by (11), (16), (37) and (38), it is easy to check that

$$v_3 = 2|A_0(\mathrm{e}^{\mathrm{j}\omega})|^4 \left\{ \sum_{i=1}^{n} |\gamma_i(\mathrm{e}^{\mathrm{j}\omega})|^2 + \mathrm{Re}\left\{ \sum_{i=1}^{n-2\rho} [\gamma_i(\mathrm{e}^{\mathrm{j}\omega})]^2 + \sum_{i=1}^{\rho} C(\beta_i)\Pi_{r-2(\rho-i+1)}(\omega) \right\}\right\}$$

$$= 2|A_0(\mathrm{e}^{\mathrm{j}\omega})|^4 \left\{ (n-r) + \sum_{k=1}^{r} |b_k(\mathrm{e}^{\mathrm{j}\omega})|^2 + \sum_{p=1}^{n-r} \cos(2p\omega) + \mathrm{Re}\left\{ \mathrm{e}^{-2(n-r)\omega j} \sum_{k=1}^{r-2\rho} [b_k(\mathrm{e}^{\mathrm{j}\omega})]^2 + \sum_{i=1}^{\rho} C(\beta_i)\Pi_{r-2(\rho-i+1)}(\omega) \right\}\right\}$$

$$= 2|A_0(\mathrm{e}^{\mathrm{j}\omega})|^4 \left\{ (n-r) + \sum_{k=1}^{r} \frac{1-|\alpha_k|^2}{|\mathrm{e}^{\mathrm{j}\omega}-\alpha_k|^2} + \sum_{p=1}^{n-r} \cos(2p\omega) + \mathrm{Re}\left\{ \sum_{k=1}^{r-2\rho} \frac{1-|\alpha_k|^2}{(\mathrm{e}^{\mathrm{j}\omega}-\alpha_k)^2}\Pi_{k-1}(\omega) + \sum_{i=1}^{\rho} C(\beta_i)\Pi_{r-2(\rho-i+1)}(\omega) \right\}\right\},$$

which is (35).

Now, again apply Lemma 2.1 with the function $g(z,\lambda) = \lambda/z$, $z = |\hat{A}|^2$ to our just established asymptotic distribution of $|\hat{A}|^2$ and using (5d) together with asymptotic independence between $\hat{\theta}$ and $\hat{\lambda}$, and the theorem follows. □

4. Discussion

The variance of the asymptotic distribution of the spectrum estimate is according to the theorem given by

$$\Lambda_n(\omega) = \frac{\mu}{\lambda^2}\Phi_y^2(\omega) + 2\Phi_y^2(\omega)S_n(\omega, A_0), \tag{39}$$

where S_n is defined by (34).

Moreover, it follows from (34) that S_n is of the form

$$S_n(\omega, A_0) = n + g_n(\omega) + f_{r,n}(\omega), \tag{40}$$

where,

$$g_n(\omega) = \sum_{p=1}^{n} \cos(2p\omega) = \begin{cases} \mathrm{Re}\left\{ \dfrac{\mathrm{e}^{2\mathrm{j}n\omega}-1}{\mathrm{e}^{-2\mathrm{j}\omega}-1} \right\} & \text{if } \omega \neq 0, \pi, \\ n & \text{else.} \end{cases} \tag{41}$$

The term $f_{r,n}$ is $-r - \sum_{p=n-r+1}^{n} \cos(2p\omega)$ plus the three last terms of (34). These sums contain no more than $3r$ terms together, and $f_{r,n}$ depends on n only via

$\sum_{p=n-r+1}^{n}\cos(2p\omega)$ and $\Pi_k(\omega)$. Obviously, $|\sum_{p=n-r+1}^{n}\cos(2p\omega)| < r$. Moreover, $|\Pi_k(\omega)|$ is n-independent, so it follows that $f_{r,n}$ is bounded by an n-independent constant, which means that

$$|S_n(\omega) - n| < C, \quad \omega \neq 0, \pi; \; C \text{ independent of } n.$$

For $\omega = 0$ or π, n should be replaced by $2n$. This means that for fixed r we have the limits

$$\lim_{n\to\infty} \frac{1}{n} S_n(\omega) = \begin{cases} 1 & \text{if } \omega \neq 0, \pi, \\ 2 & \text{else.} \end{cases} \tag{42}$$

So the classical result (7) is re-established. It also shows that the convergence rate in (42) is always like $1/n$.

It is also clear from the expressions that the convergence will depend on the pole location of the true AR-process A_0. For example, if all poles are in the origin, i.e., $\alpha_k \equiv 0$, it follows that the second term of (34) will be r and it can be verified that the sum of the fourth and fifth terms will be $\sum_{p=n-r+1}^{n}\cos(2p\omega)$. Generally speaking, poles close to the origin ($\alpha_k \approx 0$) will make $\frac{1}{n}S_n$ deviate less from its limit as $n \to \infty$.

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